

# Effective Cartan-Tanaka connections for $\mathcal{C}^6$ -smooth strongly pseudoconvex hypersurfaces $M^3 \subset \mathbb{C}^2$

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## Abstract

Explicit Cartan-Tanaka curvatures, the vanishing of which characterizes sphericity, are provided in terms of the 6-th order jet of a graphing function for a  $\mathcal{C}^6$  strongly pseudoconvex hypersurface  $M^3 \subset \mathbb{C}^2$ . *To cite this article:*

## Résumé

**Connections de Cartan-Tanaka effectives pour les hypersurfaces strictement pseudoconvexes  $M^3 \subset \mathbb{C}^2$  de classe  $\mathcal{C}^6$ .** Des courbures de Cartan-Tanaka explicites, dont l'annulation identique caractérise la sphéricité, sont fournies en termes du jet d'ordre 6 d'une fonction graphante pour une hypersurface  $M^3 \subset \mathbb{C}^2$  de classe  $\mathcal{C}^6$  strictement pseudoconvexe. *Pour citer cet article :*

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## 1. Cartan connection, curvature function and second cohomology of Lie algebras

**Definition 1.1** ([12,4]) *Let  $G$  be a real Lie group with a closed subgroup  $H$ , and let  $\mathfrak{g}$  and  $\mathfrak{h}$  be the corresponding real Lie algebras. A Cartan geometry of type  $(G, H)$  on a  $\mathcal{C}^\infty$  manifold  $M$  is a principal  $H$ -bundle  $\pi: \mathcal{P} \rightarrow M$  together with a  $\mathfrak{g}$ -valued 'Cartan connection' 1-form  $\omega: T\mathcal{P} \rightarrow \mathfrak{g}$  satisfying:*

- (i)  $\omega_p: T_p\mathcal{P} \rightarrow \mathfrak{g}$  is an isomorphism at every  $p \in \mathcal{P}$ ;
- (ii) if  $R_h(p) := ph$  is the right translation on  $\mathcal{G}$  by an  $h \in H$ , then  $R_h^*\omega = \text{Ad}(h^{-1}) \circ \omega$ ;
- (iii)  $\omega(H^\dagger) = \mathfrak{h}$  for every  $\mathfrak{h} \in \mathfrak{h}$ , where  $H^\dagger|_p := \frac{d}{dt}|_0 (R_{\exp(th)}(p))$  is the left-invariant vector field on  $\mathcal{G}$  associated to  $\mathfrak{h}$ .

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Since the associated curvature 2-form  $\Omega(X, Y) := d\omega(X, Y) + [\omega(X), \omega(Y)]_{\mathfrak{g}}$ , with  $X, Y \in \Gamma(T\mathcal{P})$ , vanishes if either  $X$  or  $Y$  is vertical ([12]),  $\Omega$  is fully represented by the *curvature function*  $\kappa \in \mathcal{C}^\infty(\mathcal{P}, \Lambda^2(\mathfrak{g}^*/\mathfrak{h}^*) \otimes \mathfrak{g})$  which sends a point  $p \in \mathcal{P}$  to the map  $\kappa(p): (\mathfrak{g}/\mathfrak{h}) \wedge (\mathfrak{g}/\mathfrak{h}) \rightarrow \mathfrak{g}$  defined by:

$$(x' \bmod \mathfrak{h}) \wedge (x'' \bmod \mathfrak{h}) \mapsto -\Omega_p(\omega_p^{-1}(x'), \omega_p^{-1}(x'')) = -[x', x'']_{\mathfrak{g}} + \omega_p([\widehat{X}', \widehat{X}'']),$$

where  $\widehat{X} := \omega^{-1}(x)$  is the *constant field* on  $\mathcal{P}$  associated to an  $x \in \mathfrak{g}$ . Denote then  $r := \dim_{\mathbb{R}} \mathfrak{g}$ ,  $n := \dim_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$  whence  $n - r = \dim_{\mathbb{R}} \mathfrak{h}$  and suppose  $r \geq 2$ ,  $n \geq 1$ ,  $n - r \geq 1$  so that  $\mathfrak{g}$ ,  $\mathfrak{g}/\mathfrak{h}$  and  $\mathfrak{h}$  are all nonzero. Picking an adapted basis  $(x_k)_{1 \leq k \leq r}$  with  $\mathfrak{g} = \text{Span}_{\mathbb{R}}(x_1, \dots, x_n, x_{n+1}, \dots, x_r)$  and  $\mathfrak{h} = \text{Span}_{\mathbb{R}}(x_{n+1}, \dots, x_r)$ , one may expand the curvature function by means of  $\mathbb{R}$ -valued components:

$$\kappa(p) = \sum_{1 \leq i_1 < i_2 \leq n} \sum_{k=1}^r \kappa_{i_1, i_2}^k(p) x_{i_1}^* \wedge x_{i_2}^* \otimes x_k.$$

**Lemma 1.2** ([2]) *For any field  $Y^\dagger = \frac{d}{dt}|_0 R_{\exp(ty)}$  on  $\mathcal{P}$  associated to an arbitrary  $y \in \mathfrak{h}$ , one has:*

$$(Y^\dagger \kappa)(p)(x', x'') = -[y, \kappa(p)(x', x'')]_{\mathfrak{g}} + \kappa(p)([y, x']_{\mathfrak{g}}, x'') + \kappa(p)(x', [y, x'']_{\mathfrak{g}}). \quad \square$$

For any  $k \in \mathbb{N}$ , consider  $k$ -cochains  $\mathcal{C}^k := \Lambda^k(\mathfrak{g}^*/\mathfrak{h}^*) \otimes \mathfrak{g}$  and define the *differential*  $\partial^k: \mathcal{C}^k \rightarrow \mathcal{C}^{k+1}$  by:

$$(\partial^k \Phi)(z_0, z_1, \dots, z_k) := \sum_{i=0}^k (-1)^i [z_i, \Phi(z_0, \dots, \widehat{z}_i, \dots, z_k)]_{\mathfrak{g}} + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \Phi([z_i, z_j]_{\mathfrak{g}}, z_0, \dots, \widehat{z}_i, \dots, \widehat{z}_j, \dots, z_k).$$

Especially for  $k = 2$ , the cohomology spaces  $\mathcal{H}^k := \ker(\partial^k)/\text{im}(\partial^{k-1})$  encode deformations of Lie algebras and are useful for Cartan connections ([13,4,6]), cf. [1] for an algorithm using Gröbner bases.

**Lemma 1.3** (Bianchi identity [4,6,2]) *For any three  $x', x'', x''' \in \mathfrak{g}$ , one has at every point  $p \in \mathcal{P}$ :*

$$0 = (\partial^2 \kappa)(p)(x', x'', x''') + \sum_{\text{cycl}} \kappa(p)(\kappa(p)(x', x''), x''') + \sum_{\text{cycl}} (\widehat{X}'(\kappa))(p)(x'', x'''). \quad \square$$

When  $\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \dots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_\nu$  is graded as in Tanaka's theory, with  $[\mathfrak{g}_{\lambda_1}, \mathfrak{g}_{\lambda_2}]_{\mathfrak{g}} \subset \mathfrak{g}_{\lambda_1 + \lambda_2}$  for any  $\lambda_1, \lambda_2 \in \mathbb{Z}$  and when  $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_\nu$ , cochains enjoy a natural grading, the second cohomology is graded too:  $\mathcal{H}^2 = \oplus_{h \in \mathbb{Z}} \mathcal{H}_{[h]}^2$ , and the graded Bianchi identities ([6,2]):

$$\partial_{[h]}^2(\kappa_{[h]})(x', x'', x''') = - \sum_{\text{cycl}} \sum_{h'=1}^{h-1} \left( \kappa_{[h-h']}(\text{proj}_{\mathfrak{g}/\mathfrak{h}}(\kappa_{[h']}(x', x'')), x''') \right) - \sum_{\text{cycl}} (\widehat{X}' \kappa_{[h+|x'|]})(x'', x''')$$

show that the lowest order nonvanishing curvature must be  $\partial$ -closed, and more generally, any homogeneous curvature component is determined by the lower components up to a  $\partial$ -closed component.

## 2. Geometry-preserving deformations of the Heisenberg sphere $\mathbb{H}^3 \subset \mathbb{C}^2$

Let now  $M^3 \subset \mathbb{C}^2$  be a *local* strongly pseudoconvex  $\mathcal{C}^6$ -smooth real 3-dimensional hypersurface, represented in coordinates  $(z, w) = (x + iy, u + iv)$  as the graph (for background, see [7,8,9,10,11]):

$$v = \varphi(x, y, u) = x^2 + y^2 + \mathcal{O}(3)$$

of a certain real-valued  $\mathcal{C}^6$  function  $\varphi$  defined in a neighborhood of the origin in  $\mathbb{R}^3$ . Its complex tangent bundle  $T^c M = \text{Re } T^{0,1} M$  is generated by the two vector fields:

$$H_1 := \frac{\partial}{\partial x} + \left( \frac{\varphi_y - \varphi_x \varphi_u}{1 + \varphi_u^2} \right) \frac{\partial}{\partial u} \quad \text{and} \quad H_2 := \frac{\partial}{\partial y} + \left( \frac{-\varphi_x - \varphi_y \varphi_u}{1 + \varphi_u^2} \right) \frac{\partial}{\partial u},$$

which make a frame joint with the third, Levi form-type Lie-bracket:

$$T := \frac{1}{4} [H_1, H_2] = \left( \frac{1}{4} \frac{1}{(1+\varphi_u^2)^2} \left\{ -\varphi_{xx} - \varphi_{yy} - 2\varphi_y \varphi_{xu} - \varphi_x^2 \varphi_{uu} + 2\varphi_x \varphi_{yu} - \varphi_y^2 \varphi_{uu} + \right. \right. \\ \left. \left. + 2\varphi_y \varphi_u \varphi_{yu} + 2\varphi_x \varphi_u \varphi_{xu} - \varphi_u^2 \varphi_{xx} - \varphi_u^2 \varphi_{yy} \right\} \right) \frac{\partial}{\partial u}.$$

Such  $M^3$ 's are *geometry-preserving* deformations of the *Heisenberg sphere*  $\mathbb{H}^3: v = x^2 + y^2$ . It is known that the Lie algebra  $\mathfrak{hol}(\mathbb{H}^3) := \{X = Z(z, w) \frac{\partial}{\partial z} + W(z, w) \frac{\partial}{\partial w} : X + \bar{X} \text{ tangent to } \mathbb{H}^3\}$  of infinitesimal CR automorphisms of the Heisenberg sphere  $\mathbb{H}^3$  in  $\mathbb{C}^2$  is 8-dimensional and generated by:

$$\begin{aligned} \mathbb{T} &:= \partial_w, & \mathbb{H}_1 &:= \partial_z + 2iz \partial_w, & \mathbb{H}_2 &:= i \partial_z + 2z \partial_w, & \mathbb{D} &:= z \partial_z + 2w \partial_w, & \mathbb{R} &:= iz \partial_z, \\ \mathbb{l}_1 &:= (w + 2iz^2) \partial_z + 2izw \partial_w, & \mathbb{l}_2 &:= (iw + 2z^2) \partial_z + 2zw \partial_w & \mathbb{J} &:= zw \partial_z + w^2 \partial_w. \end{aligned}$$

Recently, Ezhov, McLaughlin and Schmalz published in the *Notices of the AMS* an expository article [6] in which they reconstruct — within Tanaka's framework and assuming that  $M$  is  $\mathcal{C}^\omega$  — Cartan's connection ([5]) valued the eight-dimensional *abstract* real Lie algebra:

$$\mathfrak{g} := \mathbb{R} \mathfrak{t} \oplus \mathbb{R} \mathfrak{h}_1 \oplus \mathbb{R} \mathfrak{h}_2 \oplus \mathbb{R} \mathfrak{d} \oplus \mathbb{R} \mathfrak{r} \oplus \mathbb{R} \mathfrak{i}_1 \oplus \mathbb{R} \mathfrak{i}_2 \oplus \mathbb{R} \mathfrak{j} \quad (\text{with } \mathfrak{h} := \mathbb{R} \mathfrak{d} \oplus \mathbb{R} \mathfrak{r} \oplus \mathbb{R} \mathfrak{i}_1 \oplus \mathbb{R} \mathfrak{i}_2 \oplus \mathbb{R} \mathfrak{j})$$

spanned by some eight abstract vectors enjoying the same commutator table as  $\mathbb{T}, \dots, \mathbb{J}$ . A natural Tanaka grading is:  $\mathfrak{g}_{-2} = \mathbb{R} \mathfrak{t}$ ;  $\mathfrak{g}_{-1} = \mathbb{R} \mathfrak{h}_1 \oplus \mathbb{R} \mathfrak{h}_2$ ;  $\mathfrak{g}_0 = \mathbb{R} \mathfrak{d} \oplus \mathbb{R} \mathfrak{r}$ ;  $\mathfrak{g}_1 = \mathbb{R} \mathfrak{i}_1 \oplus \mathbb{R} \mathfrak{i}_2$ ;  $\mathfrak{g}_2 = \mathbb{R} \mathfrak{j}$ . By performing the above choice  $\{H_1, H_2, T\}$  of an initial frame for  $TM$  which is explicit in terms of the graphing function  $\varphi(x, y, u)$ , we deviate from the initial normalization made in [6] (with a more geometric-minded approach), since our computational objective is to provide a Cartan-Tanaka connection all elements of which are completely effective in terms of  $\varphi(x, y, u)$  — assuming only  $\mathcal{C}^6$ -smoothness of  $M$ .

Call  $\Upsilon$  the numerator of  $T = \frac{1}{4} [H_1, H_2] = \frac{1}{4} \frac{\Upsilon}{\Delta^2} \frac{\partial}{\partial u}$ , allow the two notational coincidences:  $x_1 \equiv x$ ,  $x_2 \equiv y$ ; introduce the two length-three brackets:

$$[H_i, T] = \frac{1}{4} [H_i, [H_1, H_2]] =: \Phi_i T \quad (i=1, 2),$$

which are both multiples of  $T$  by means of two functions  $\Phi_i := \frac{A_i}{\Delta^2 \Upsilon}$ ; lastly, introduce furthermore the  $H_k$ -iterated derivatives of the functions  $\Phi_i$  up to order 3, where  $i, k_1, k_2, k_3 = 1, 2$ :

$$H_{k_1}(\Phi_i) = \frac{A_{i, k_1}}{\Delta^4 \Upsilon^2} \quad H_{k_2}(H_{k_1}(\Phi_i)) = \frac{A_{i, k_1, k_2}}{\Delta^6 \Upsilon^3}, \quad H_{k_3}(H_{k_2}(H_{k_1}(\Phi_i))) = \frac{A_{i, k_1, k_2, k_3}}{\Delta^8 \Upsilon^4}.$$

**Proposition 2.1** ([2]) *All the numerators appearing above are explicitly given by:*

$$\begin{aligned} A_i &:= \Delta^2 \Upsilon_{x_i} + \Delta(-2 \Delta_{x_i} \Upsilon + \Lambda_i \Upsilon_u - \Upsilon \Lambda_{i, u}) - \Lambda_i \Upsilon \Delta_u, \\ A_{i, k_1} &:= \Delta^2(\Upsilon A_{i, x_{k_1}} - \Upsilon_{x_{k_1}} A_i) + \Delta(-2 \Delta_{x_{k_1}} \Upsilon A_i + \Upsilon \Lambda_{k_1} A_{i, u} - \Upsilon_u \Lambda_{k_1} A_i) - 2 \Delta_u \Upsilon \Lambda_{k_1} A_i, \\ A_{i, k_1, k_2} &:= \Delta^2(\Upsilon A_{i, k_1, x_{k_2}} - 2 \Upsilon_{x_{k_2}} A_{i, k_1}) + \Delta(-3 \Delta_{x_{k_2}} \Upsilon A_{i, k_1} + \Upsilon \Lambda_{k_2} A_{i, k_1, u} - 2 \Upsilon_u \Lambda_{k_2} A_{i, k_1}) - 3 \Delta_u \Upsilon \Lambda_{k_2} A_{i, k_1}, \\ A_{i, k_1, k_2, k_3} &:= \Delta^2(\Upsilon A_{i, k_1, k_2, x_{k_3}} - \Upsilon_{x_{k_3}} A_{i, k_1, k_2}) + \Delta(-6 \Delta_{x_{k_3}} \Upsilon A_{i, k_1, k_2} + \Upsilon \Lambda_{k_3} A_{i, k_1, k_2, u} - 3 \Upsilon_u \Lambda_{k_3} A_{i, k_1, k_2}) - 6 \Delta_u \Upsilon \Lambda_{k_3} A_{i, k_1, k_2}. \end{aligned}$$

Furthermore, these iterated derivatives identically satisfy  $H_2(\Phi_1) \equiv H_1(\Phi_2)$  and:

$$\begin{aligned} 0 &\equiv -H_1(H_2(H_1(\Phi_2))) + 2H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - \Phi_2 H_1(H_2(\Phi_1)) + \Phi_2 H_2(H_1(\Phi_1)), \\ 0 &\equiv -H_2(H_1(H_1(\Phi_2))) + 2H_1(H_2(H_1(\Phi_2))) - H_1(H_1(H_2(\Phi_2))) - \Phi_1 H_2(H_1(\Phi_2)) + \Phi_1 H_1(H_2(\Phi_2)), \\ 0 &\equiv -H_1(H_1(H_1(\Phi_2))) + 2H_1(H_2(H_1(\Phi_1))) - H_2(H_1(H_1(\Phi_1))) + \Phi_1 H_1(H_1(\Phi_2)) - \Phi_1 H_2(H_1(\Phi_1)), \\ 0 &\equiv -H_2(H_2(H_1(\Phi_2))) + 2H_2(H_1(H_2(\Phi_2))) - H_1(H_2(H_2(\Phi_2))) + \Phi_2 H_2(H_1(\Phi_2)) - \Phi_2 H_1(H_2(\Phi_2)). \end{aligned}$$

### 3. Explicit Cartan-Tanaka connection

**Theorem 3.1** ([2]) *Associated to such an  $M^3 \subset \mathbb{C}^2$ , there is a unique  $\mathfrak{g}$ -valued Cartan connection which is normal and regular in the sense of Tanaka. Its curvature function reduces to:*

$$\begin{aligned}\kappa(p) = & \kappa_{i_1}^{h_1 t}(p) \mathbf{h}_1^* \wedge \mathbf{t}^* \otimes \mathbf{i}_1 + \kappa_{i_2}^{h_1 t}(p) \mathbf{h}_1^* \wedge \mathbf{t}^* \otimes \mathbf{i}_2 + \kappa_{i_1}^{h_2 t}(p) \mathbf{h}_2^* \wedge \mathbf{t}^* \otimes \mathbf{i}_1 + \\ & + \kappa_{i_2}^{h_2 t}(p) \mathbf{h}_2^* \wedge \mathbf{t}^* \otimes \mathbf{i}_2 + \kappa_j^{h_1 t}(p) \mathbf{h}_1^* \wedge \mathbf{t}^* \otimes \mathbf{j} + \kappa_j^{h_2 t}(p) \mathbf{h}_2^* \wedge \mathbf{t}^* \otimes \mathbf{j},\end{aligned}$$

where the two main curvature coefficients, having homogeneity 4, are of the form:

$$\kappa_{i_1}^{h_1 t}(p) = -\Delta_1 c^4 - 2\Delta_4 c^3 d - 2\Delta_4 c d^3 + \Delta_1 d^4 \quad \text{and} \quad \kappa_{i_2}^{h_1 t}(p) = -\Delta_4 c^4 + 2\Delta_1 c^3 d + 2\Delta_1 c d^3 + \Delta_4 d^4,$$

in which the two functions  $\Delta_1$  and  $\Delta_4$  of only the three variables  $(x, y, u)$  are explicitly given by:

$$\begin{aligned}\Delta_1 = & \frac{1}{384} [H_1(H_1(H_1(\Phi_1))) - H_2(H_2(H_2(\Phi_2))) + 11 H_1(H_2(H_1(\Phi_2))) - 11 H_2(H_1(H_2(\Phi_1))) + 6 \Phi_2 H_2(H_1(\Phi_1)) - 6 \Phi_1 H_1(H_2(\Phi_2)) \\ & - 3 \Phi_2 H_1(H_1(\Phi_2)) + 3 \Phi_1 H_2(H_2(\Phi_1)) - 3 \Phi_1 H_1(H_1(\Phi_1)) + 3 \Phi_2 H_2(H_2(\Phi_2)) - 2 \Phi_1 H_1(\Phi_1) + 2 \Phi_2 H_2(\Phi_2) - \\ & - 2(\Phi_2)^2 H_1(\Phi_1) + 2(\Phi_1)^2 H_2(\Phi_2) - 2(\Phi_2)^2 H_2(\Phi_2) + 2(\Phi_1)^2 H_1(\Phi_1)],\end{aligned}$$

$$\begin{aligned}\Delta_4 = & \frac{1}{384} [-3 H_2(H_1(H_2(\Phi_2))) - 3 H_1(H_2(H_1(\Phi_1))) + 5 H_1(H_2(H_2(\Phi_2))) + 5 H_2(H_1(H_1(\Phi_1))) + \\ & + 4 \Phi_1 H_1(H_1(\Phi_2)) + 4 \Phi_2 H_2(H_1(\Phi_2)) - 3 \Phi_2 H_1(H_1(\Phi_1)) - 3 \Phi_1 H_2(H_2(\Phi_2)) - \\ & - 7 \Phi_2 H_1(H_2(\Phi_2)) - 7 \Phi_1 H_2(H_1(\Phi_1)) - 2 H_1(\Phi_1) H_1(\Phi_2) - 2 H_2(\Phi_2) H_2(\Phi_1) + \\ & + 4 \Phi_1 \Phi_2 H_1(\Phi_1) + 4 \Phi_1 \Phi_2 H_2(\Phi_2)],\end{aligned}$$

and where the remaining four secondary curvature coefficients are given by:

$$\kappa_{i_1}^{h_2 t} = \kappa_{i_2}^{h_1 t}, \quad \kappa_{i_2}^{h_2 t} = -\kappa_{i_1}^{h_1 t}, \quad \kappa_j^{h_1 t} = \widehat{H}_1(\kappa_{i_2}^{h_2 t}) - \widehat{H}_2(\kappa_{i_2}^{h_1 t}), \quad \kappa_j^{h_2 t} = -\widehat{H}_1(\kappa_{i_1}^{h_2 t}) + \widehat{H}_2(\kappa_{i_1}^{h_1 t}).$$

**Corollary 3.2** *A  $\mathcal{C}^6$ -smooth strongly pseudoconvex local hypersurface  $M^3 \subset \mathbb{C}^2$  is biholomorphic to  $\mathbb{H}^3$ , namely is spherical, if and only if  $0 \equiv \Delta_1 \equiv \Delta_4$ , identically as functions of  $(x, y, u)$ .*

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