A Gröbner-Bases Algorithm for the Computation of the Cohomology of Lie (Super) Algebras

Mansour Aghasi, Benyamin M.-Alizadeh, Joël Merker, Masoud Sabzevari

A GRÖBNER-BASES ALGORITHM FOR THE COMPUTATION OF THE COHOMOLOGY OF LIE (SUPER) ALGEBRAS

MANSOUR AGHASI, BENYAMIN M.-ALIZADEH, JOËL MERKER, AND MASOUD SABZEVARI

ABSTRACT. We present an effective algorithm for computing the standard cohomology spaces of finitely generated Lie (super) algebras over a field $\mathbb{K}$ of characteristic zero. In order to reach explicit representatives of some generators of the quotient space $\mathscr{Z}^k \big/ \mathscr{B}^k$ of cocycles $\mathscr{Z}^k$ modulo coboundaries $\mathscr{B}^k$, we apply Gröbner bases techniques (in the appropriate linear setting) and take advantage of their strength. Moreover, when the considered Lie (super) algebras enjoy a grading — a case which often happens both in representation theory and in differential geometry —, all cohomology spaces $\mathscr{Z}^k \big/ \mathscr{B}^k$ naturally split up as direct sums of smaller subspaces, and this enables us, for higher dimensional Lie (super) algebras, to improve the computer speed of calculations. Lastly, we implement our algorithm in the MAPLE software and evaluate its performances via some examples, most of which have several applications in the theory of Cartan-Tanaka connections.

1. INTRODUCTION

The concept of cohomology group — one of the central concepts in contemporary science — possesses established applications in several areas of pure mathematics, for instance: deformation of Lie algebras ([11]); analytic partial differential equations; global foliation theory; combinatorics (Mcdonald identities); invariant differential operators; cobordism theory; infinite-dimensional Lie algebras ([10]); exterior differential systems; Cartan-Tanaka theory of connections ([5, 1, 2, 19]); *etc.* Moreover, cohomology groups also have applications in quantum physics; for quasi-invariancy of certain Lagrangians; in the Wess-Zumino-Novikov-Witten model (*cf.* [3]); when one reinterprets general relativity by means of $\mathfrak{so}(3,1)$-valued connections; *etc.* It therefore turns out to be worthwhile to set up appropriate efficient algorithms for the computation of Lie (super) algebra cohomologies, granted that calculations quickly become hard by hand.

Recently, a few articles have been published in this direction. Kornyak [14, 15] devised an algorithm and implemented it in the C program. Moreover, Grozman, Leites, Post and Von Hijligenberg ([12, 17, 20]) prepared some packages for computing Lie (super) algebra cohomologies in REDUCE and in MATHEMATICA. In the present article, motivated by the specific objective of developing the construction of *effective* Cartan-Tanaka connections that are valued in Lie algebras which are *not* semi-simple (*see* [5, 1, 2, 19] for some instances of that research program

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and also [8] in the parabolic/simple case), our main aim is to set up an alternative algorithm and to implement it in the MAPLE software. We would like to employ the method of *Gröbner bases*, a modern, effective and widespread tool in computational mathematics. Of course, the continued regular progresses in Gröbner bases algorithms enrich *de facto* any algorithm that is built on them. For convenience and self-contentness, a short reminder of Gröbner bases concepts will be given in Section 2. But before that, let us present a brief description of the definitions, notations and formulas in Lie super algebras, and let us introduce their cohomology groups, precisely.

A *Lie super algebra* over a field $\mathbb{K}$ of characteristic zero is a $(\mathbb{Z}/2\mathbb{Z})$-graded algebra which is a direct sum (as a vector space):

$$\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$$

of two subspaces $\mathfrak{g}_{\overline{0}}$ and $\mathfrak{g}_{\overline{1}}$ subjected to the following structural properties. An element $\mathsf{x} \in \mathfrak{g}$ is *homogeneous* if either $\mathsf{x} \in \mathfrak{g}_{\overline{0}}$ or $\mathsf{x} \in \mathfrak{g}_{\overline{1}}$, and in this case, its *weight* $|\mathsf{x}|$ is defined to be 0 or 1, accordingly (the elements of $\mathfrak{g}_{\overline{0}}$ and of $\mathfrak{g}_{\overline{1}}$ are called *even* and *odd*, respectively). The algebra structure is a *degree-zero* bilinear Lie bracket $[\cdot,\cdot]: \ \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which is graded, namely it satisfies:

$$\big[\mathfrak{g}_{\overline{i}}, \mathfrak{g}_{\overline{j}}\big] \subseteq \mathfrak{g}_{\overline{i+j}},$$

for any $i, j = 0, 1$ where $\overline{i+j} = i + j \mod 2$. The bracket also satisfies, for arbitrary homogeneous elements $\mathsf{x}, \mathsf{y}, \mathsf{z}$ belonging either to $\mathfrak{g}_{\overline{0}}$ or to $\mathfrak{g}_{\overline{1}}$:

$$[\mathsf{x}, \mathsf{y}] = -(-1)^{|\mathsf{x}||\mathsf{y}|}[\mathsf{y}, \mathsf{x}] \qquad \text{(super skew-symmetry)},$$

$$\big[\mathsf{x}, [\mathsf{y}, \mathsf{z}]\big] = \big[[\mathsf{x}, \mathsf{y}], \mathsf{z}\big] + (-1)^{|\mathsf{x}||\mathsf{y}|}\big[\mathsf{y}, [\mathsf{x}, \mathsf{z}]\big] \qquad \text{(super Jacobi identity)},$$

these relations being then extended by $\mathbb{K}$-linearity to all elements of $\mathfrak{g}$. In differentialo-geometric applications ([5, 8, 1, 2, 19]), the field $\mathbb{K}$ of characteristic zero is usually assumed to be either just $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$, plainly.

A $\mathfrak{g}$-*module* V is a vector space over the same field $\mathbb{K}$ together with a bilinear map (denoted shortly with a dot) $\cdot\colon \mathfrak{g} \times V \to V$ having the property:

$$[\mathsf{x}, \mathsf{y}] \cdot v = \mathsf{x} \cdot (\mathsf{y} \cdot v) - (-1)^{|\mathsf{x}||\mathsf{y}|}\mathsf{y} \cdot (\mathsf{x} \cdot v),$$

for any two homogeneous $\mathsf{x}, \mathsf{y} \in \mathfrak{g}_{\overline{i}}$, $i = 0, 1$, and any $v \in V$. One of the most important instances of such $\mathfrak{g}$-modules occurs when $\mathfrak{g}$ happens to be a Lie (super) subalgebra of a certain larger Lie (super) algebra $\mathfrak{h} =: V$, with the bilinear map $\cdot\colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ being just precisely the Lie bracket of $\mathfrak{h}$, of course.

Thus, let $\mathfrak{g}$ be an m-dimensional Lie super algebra and let V be a $\mathfrak{g}$-module. For any integer $k \geqslant 0$, the space $\mathscr{C}^k(\mathfrak{g}, V)$ of k-*cochains* consists of the space of k-multilinear maps:

$$\Phi\colon \ \mathfrak{g}^k \longrightarrow V,$$

where $\mathfrak{g}^k = \mathfrak{g} \times \cdots \times \mathfrak{g}$ (k times, with $\mathfrak{g}^0 = \{0\}$ naturally), that are *super skew-symmetric* in the sense that:

$$\Phi\big(\mathsf{z}_1, \ldots, \mathsf{z}_i, \mathsf{z}_{i+1}, \ldots, \mathsf{z}_k\big) = -(-1)^{|\mathsf{z}_i||\mathsf{z}_{i+1}|}\, \Phi\big(\mathsf{z}_1, \ldots, \mathsf{z}_{i+1}, \mathsf{z}_i, \ldots, \mathsf{z}_k\big),$$

for any homogeneous arguments. Then for any integer $k \geqslant 0$, there is a fundamental linear *differential operator:*

$$\partial^k\colon \ \ \mathscr{C}^k\big(\mathfrak{g}, V\big) \longrightarrow \mathscr{C}^{k+1}\big(\mathfrak{g}, V\big),$$

mapping a k-cochain Φ uniquely to a $(k+1)$-cochain $\partial^k\Phi$ that acts as follows (*see* [10, 13]) on any collection of $k+1$ homogeneous elements $\mathsf{e}_0, \dots, \mathsf{e}_p \in \mathfrak{g}_{\bar{0}}$, and $\mathsf{o}_{p+1}, \dots, \mathsf{o}_k \in \mathfrak{g}_{\bar{1}}$:

$$
\begin{aligned}
&(\partial^k\Phi)\big(\mathsf{e}_0, \dots, \mathsf{e}_p,\, \mathsf{o}_{p+1}, \dots, \mathsf{o}_k\big) := \\
&:= \sum_{i=0}^{p} (-1)^{i+1}\, \mathsf{e}_i \cdot \Phi\big(\mathsf{e}_0, \dots, \widehat{\mathsf{e}}_i, \dots, \mathsf{e}_p,\, \mathsf{o}_{p+1}, \dots, \mathsf{o}_k\big) + \\
&+ \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j+1}\, \Phi\big([\mathsf{e}_i, \mathsf{e}_j], \mathsf{e}_0, \dots, \widehat{\mathsf{e}}_i, \dots, \widehat{\mathsf{e}}_j, \dots, \mathsf{e}_p,\, \mathsf{o}_{p+1}, \dots, \mathsf{o}_k\big) + \\
&+ \sum_{i=0}^{p} \sum_{j=p+1}^{k} (-1)^{i}\, \Phi\big(\mathsf{e}_0, \dots, \widehat{\mathsf{e}}_i, \dots, \mathsf{e}_p, [\mathsf{e}_i, \mathsf{o}_j],\, \mathsf{o}_{p+1}, \dots, \widehat{\mathsf{o}}_j, \dots, \mathsf{o}_k\big) + \\
&+ \sum_{p+1 \leqslant i < j \leqslant k} \Phi\big([\mathsf{o}_i, \mathsf{o}_j], \mathsf{e}_0, \dots, \mathsf{e}_p,\, \mathsf{o}_{p+1}, \dots, \widehat{\mathsf{o}}_i, \dots, \widehat{\mathsf{o}}_j, \dots, \mathsf{o}_k\big) + \\
&+ (-1)^p \sum_{i=p+1}^{k} \mathsf{o}_i \cdot \Phi\big(\mathsf{e}_0, \dots, \dots, \mathsf{e}_p,\, \mathsf{o}_{p+1}, \dots, \widehat{\mathsf{o}}_i, \dots, \mathsf{o}_k\big),
\end{aligned}
\tag{1}
$$

where as usual, $\widehat{\mathsf{z}}_l$ means removal of the term z_l (in the case of Lie algebras, comparing with some references such as [1, 3, 11, 19], there is an overall minus sign in the right-hand side). One checks ([10]) that in the case of standard Lie algebras $\mathfrak{g} \subset \mathfrak{h} = V$, only the first two lines of the above definition are non-zero, and in fact, for any $k+1$ vectors $\mathsf{z}_0, \mathsf{z}_1, \dots, \mathsf{z}_k \in \mathfrak{g}$, one has:

$$
\begin{aligned}
(\partial^k\Phi)\big(\mathsf{z}_0, \mathsf{z}_1, \dots, \mathsf{z}_k\big) &:= \sum_{i=0}^{k} (-1)^i \big[\mathsf{z}_i,\, \Phi(\mathsf{z}_0, \dots, \widehat{\mathsf{z}}_i, \dots, \mathsf{z}_k)\big] + \\
&+ \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j}\, \Phi\big([\mathsf{z}_i, \mathsf{z}_j], \mathsf{z}_0, \dots, \widehat{\mathsf{z}}_i, \dots, \widehat{\mathsf{z}}_j, \dots, \mathsf{z}_k\big).
\end{aligned}
\tag{2}
$$

In both cases, this $(k+1)$-cochain $\partial^k\Phi$ is clearly linear with respect to each argument, and furthermore, it is (super) skew-symmetric ([10]). Furthermore, one can verify that the compositions $\partial^{k+1} \circ \partial^k$ vanish for any $k \in \mathbb{N}$, hence we have the following *cochain complex*:

$$
0 \xrightarrow{\partial^0} \mathscr{C}^1 \xrightarrow{\partial^1} \mathscr{C}^2 \xrightarrow{\partial^2} \cdots \xrightarrow{\partial^{m-2}} \mathscr{C}^{m-1} \xrightarrow{\partial^{m-1}} \mathscr{C}^m \xrightarrow{\partial^m} 0. \tag{3}
$$

Based on these definitions, the *k-th cohomological space* $H^k(\mathfrak{g}, V)$ is defined to be the following quotient space:

$$
H^k\big(\mathfrak{g}, V\big) = \frac{\mathscr{Z}^k(\mathfrak{g}, V)}{\mathscr{B}^k(\mathfrak{g}, V)},
$$

where $\mathscr{Z}^k(\mathfrak{g}, V) := \ker\big(\partial^k\big)$ and $\mathscr{B}^k(\mathfrak{g}, V) := \operatorname{im}\big(\partial^{k-1}\big)$.

Within MAPLE, there exists a package entitled LieAlgebraCohomology which computes a somewhat different type of Lie algebra cohomology, called *relative cohomology*. In particular, this package computes the *De Rham cohomoloy*, quite central in differential geometry. But still, there is no package or command for computing the above-mentioned type of cohomological spaces of Lie (super) algebras, although it has several applications to, *e.g.*, the differential geometry of Cartan-Tanaka connections.

The article is divided into five sections. In Section 2, some preliminaries about Gröbner bases are reviewed. Section 3 is devoted to the main results of this paper. In Section 4 we describe our algorithm to compute the cohomological spaces of certain Lie algebras. Lastly, in Section 5 we show, with some examples, that computations naturally split up, in the case of a pair of plain Lie (sub)algebras $\mathfrak{g} \subset \mathfrak{h} = V$, when $\mathfrak{g}$ and $\mathfrak{h}$ are simultaneously graded.

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2. GRÖBNER BASES AND ELIMINATION IDEALS

The theory of Gröbner bases is a key computational tool for studying polynomial ideals. This theory was introduced and developed by Buchberger, who devised its general scheme in the early 1960's ([6, 7]). Nowadays, there exist several refined and improved algorithms that are more efficient than the original one, such as F_4, F_5, FGB, GB, G^2V and GVW, and most of them have been regularly implemented in computer algebra systems like MAPLE, MAGMA, MATHEMATICA, SINGULAR, MACAULAY2, COCOA and SAGE.

To provide a summarized description of the theory, borrowing the notation and the results to the monograph [9] of Cox, Little and O'Shea, let $\mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring in $n \geqslant 1$ variables on some arbitrary field $\mathbb{K}$ of characteristic zero and let $\mathscr{I} = \langle f_1, \dots, f_k \rangle$ be any ideal of $\mathbb{K}[x_1, \dots, x_n]$ generated by a finite number (noetherianity!) of polynomials $f_1, \dots, f_k \in \mathbb{K}[x_1, \dots, x_n]$.

Definition 2.1. A *monomial ordering* on $\mathbb{K}[x_1, \dots, x_n]$ is a binary relation $\prec$ on the set of monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in $\mathbb{K}[x_1, \dots, x_n]$ which satisfies:

- $\prec$ is a strict total ordering, namely it is transitive, asymmetric and any two monomials are comparable;
- $x^\alpha \prec x^\beta$ implies $x^\gamma x^\alpha \prec x^\gamma x^\beta$ for every monomial x^γ, $\gamma \in \mathbb{N}^n$;
- $\prec$ is a well-ordering, namely, every nonempty set of monomials has a minimal element.

For example, the usual *lexicographical* ordering, here denoted $\prec_{\mathsf{lex}}$, is a monomial ordering defined as follows ([4, 9]): if $\deg_i(m)$ denotes the degree in x_i of a monomial m, if m' and m'' are two monomials, then $m' \prec_{\mathsf{lex}} m''$ if and only if (by definition) the first nonzero entry of the vector of $\mathbb{Z}^n$:

$$\big(\deg_1(m'') - \deg_1(m'), \dots, \deg_n(m'') - \deg_n(m')\big)$$

is positive.

Let now $\prec$ be any monomial ordering on $\mathbb{K}[x_1, \dots, x_n]$. The *leading monomial* of a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ is the greatest monomial — with respect to $\prec$ — which appears in f, and we denote it by $\mathrm{LM}(f)$. Furthermore, the *leading* *coefficient* of f, written by $\mathrm{LC}(f) \in \mathbb{K}$, is the $\mathbb{K}$-coefficient of $\mathrm{LM}(f)$ in f and the *leading term* of f is the product:

$$\mathrm{LT}(f) := \mathrm{LC}(f) \cdot \mathrm{LM}(f).$$

The following theorem states a fundamental *division algorithm* in $\mathbb{K}[x_1, \ldots, x_n]$.

Theorem 2.1. ([4, 9]) *Given a fixed monomial ordering $\prec$ on $\mathbb{K}[x_1, \ldots, x_n]$, for any* ordered *$k$-tuple $(f_1, \ldots, f_k)$ of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$, every $f \in \mathbb{K}[x_1, \ldots, x_n]$ can be written as:*

$$f = a_1 f_1 + \cdots + a_k f_k + r,$$

for some $a_i, r \in \mathbb{K}[x_1, \ldots, x_n]$, with the main property that either $r = 0$ or r is a linear combination of monomials, none of which *is divisible by any $\mathrm{LT}(f_j)$, $j = 1, \ldots, k$.*

Usually, one calls r a *(one)* remainder of f on division by $(f_1, \ldots, f_k)$, because most often, it is *not* unique, and because in addition, it strongly depends on the ordering of the f_i's. This theorem, a higher-dimensional version of the standard Euclidean division algorithm valid for the one-dimensional ring $\mathbb{K}[x_1]$, is the main effective cornerstone in the field of Gröbner bases; in fact, search for higher speed concentrates mainly on improving the efficiency of division. Next, we define what is a Gröbner basis for a polynomial ideal $\mathscr{I} \subset \mathbb{K}[x_1, \ldots, x_n]$.

Definition 2.2. A finite subset $\mathtt{G} = \{g_1, \ldots, g_l\} \subset \mathscr{I}$ is called a *Gröbner basis* of $\mathscr{I}$ with respect to some fixed monomial ordering $\prec$ if the ideal generated by the leading monomials of all elements of $\mathscr{I}$ coincides with the monomial ideal generated by the $\mathrm{LT}(g_j)$, $j = 1, \ldots, l$:

$$\big\langle\, \mathrm{LT}(f) \colon f \in \mathscr{I} \,\big\rangle = \big\langle\, \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_l) \,\big\rangle.$$

Next, if $\mathtt{G} = \{g_1, \ldots, g_l\}$ is a Gröbner basis of an ideal with respect to some monomial ordering $\prec$, one proves that the remainder, on division by $\mathtt{G}$, of any $f \in \mathbb{K}[x_1, \ldots, x_n]$ is *unique*, one calls this remainder the *normal form* of f with respect to $\mathtt{G}$ and one denotes it by $\mathrm{NF}_{\mathtt{G}}(f)$, *cf.* again [4, 9]. Also, one proves that if $\mathtt{G}$ is a Gröbner basis for $\mathscr{J}$, then $\mathrm{NF}_{\mathtt{G}}(f) = 0$ if and only if $f \in \mathscr{J} = \langle\, \mathtt{G} \,\rangle$. Then the fundamental theorem of the theory is that *every* nonzero ideal $\mathscr{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ possesses at least one Gröbner basis, with (refinable) algorithms which produces such a Gröbner basis from any set of generators, by taking so-called *S-polynomials* between any two distinct generators and by applying, inductively, the division Theorem 2.1. Furthermore, if $\mathtt{G}$ is any Gröbner basis of $\mathscr{I}$, it also generates $\mathscr{I}$, hopefully. However, Gröbner bases for an ideal are not unique. Once a monomial order is chosen, reduced Gröbner bases fully insure uniqueness.

Definition 2.3. A *reduced Gröbner basis* of an ideal $\mathscr{I}$ is a Gröbner basis $\mathtt{G} = \{g_1, \ldots, g_l\}$ of $\mathscr{I}$ whose polynomials g_j are all monic such that, for any two distinct $g_{j_1}, g_{j_2} \in \mathtt{G}$, no monomial appearing in g_{j_2} is a multiple of $\mathrm{LT}(g_{j_1})$.

Then one establishes ([4, 9]) that, given a fixed monomial ordering $\prec$ on the ring $\mathbb{K}[x_1, \ldots, x_n]$, every ideal $\mathscr{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ possesses a *unique* reduced Gröbner basis.

The concept of *elimination ideal*, a natural application of Gröbner bases, will be a very useful tool for us. Consider again $\mathbb{K}[x_1, \ldots, x_n]$ and pick a (finite) subset of m, with $1 \leqslant m \leqslant n-1$, variables among the n variables $\{x_1, \ldots, x_n\}$; possibly after a permutation, these (sub)variables may of course be assumed to be just $x_1, \ldots, x_m$. Then, for any ideal $\mathscr{I} \subset \mathbb{K}[x_1, \ldots, x_m, x_{m+1}, \ldots, x_n]$, we call:

$$\mathscr{I} \cap \mathbb{K}[x_1, \ldots, x_m],$$

the *elimination ideal* of $\mathscr{I}$ with respect to the (sub)variables:

$$\big\{x_1, \ldots, x_m\big\} \subset \big\{x_1, \ldots, x_m, x_{m+1}, \ldots, x_n\big\}.$$

The following proposition provides one with a way to compute elimination ideals, using Gröbner bases, and, as a bonus, it also yields at the same time a reduced Gröbner basis for the elimination ideal.

Proposition 2.4. ([4, 9]) *Let $\prec$ be a monomial ordering on the ring $\mathbb{K}[x_1, \ldots, x_m, x_{m+1}, \ldots, x_n]$ having the property that $x_j \prec x_k$ for any $j = 1, \ldots, m$ and any $k = m+1, \ldots, n$, and let $\mathtt{G}$ be the reduced Gröbner basis of $\mathscr{I}$ with respect to $\prec$. Then $\mathtt{G} \cap \mathbb{K}[x_1, \ldots, x_m]$ is a reduced Gröbner basis for the elimination ideal $\mathscr{I} \cap \mathbb{K}[x_1, \ldots, x_m]$ with respect to $\prec$.*

3. Computation of Cohomology Spaces

Now, coming back to our goal, let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ be an m-dimensional Lie super algebra generated as a $\mathbb{K}$-vector space by p even elements $\mathsf{e}_1, \ldots \mathsf{e}_p$ and by $m-p$ odd elements $\mathsf{o}_{p+1}, \ldots \mathsf{o}_m$, and let V be an n-dimensional $\mathfrak{g}$-module generated by vectors $v_1, \ldots, v_n$, as a $\mathbb{K}$-vector space too. It is natural to divide any algorithm on the computation of Lie super algebra cohomologies into three steps:

- computation of the *space of cocycles* $\mathscr{Z}^k(\mathfrak{g}, V)$;
- computation of the *space of coboundaries* $\mathscr{B}^k(\mathfrak{g}, V)$;
- computation of the cohomology space $H^k(\mathfrak{g}, V) = \mathscr{Z}^k(\mathfrak{g}, V)\big/\mathscr{B}^k(\mathfrak{g}, V)$.

Sometimes, we shall abbreviate simply by $\mathscr{Z}^k$ the space $\mathscr{Z}^k(\mathfrak{g}, V)$, and so on. Obviously, the most substantial step of the algorithm is the third one, in which one has to compute the quotient of the two spaces obtained, at the first and second steps, by somewhat routine computations. Accordingly, we shall divide this section into three steps in which we explain the corresponding fraction of the algorithm.

3.1. **Computation of $\mathscr{Z}^{\mathbf{k}}(\mathfrak{g}, \mathbf{V})$.** At first, we have to determine a basis for the vector space $\mathscr{C}^k(\mathfrak{g}, V)$. For any $r = 0, \ldots, k$, for any $1 \leqslant i_1 < \cdots < i_r \leqslant p$, for any $p+1 \leqslant j_{r+1} \leqslant \cdots \leqslant j_k \leqslant m$ and for any $l = 1, \ldots, n$, let us denote by:

$$\Lambda_l^{(i_1, \ldots, i_r | j_{r+1}, \ldots, j_k)}$$

the basic element (map) of $\mathscr{C}^k(\mathfrak{g}, V)$ whose value on $(\mathsf{e}_{i_1}, \ldots, \mathsf{e}_{i_r}, \mathsf{o}_{j_{r+1}}, \ldots, \mathsf{o}_{j_k})$ is exactly $1 \cdot v_l$, which acts super-symmetrically and which is zero elsewhere. One verifies that the set of these maps constitutes a basis over $\mathbb{K}$ for the vector space $\mathscr{C}^k(\mathfrak{g}, V)$, hence a general k-cochain Φ naturally decomposes as a linear combination:

$$\Phi = \sum_{r=0}^{k} \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant p} \sum_{p+1 \leqslant j_{r+1} \leqslant \cdots \leqslant j_k \leqslant m} \sum_{l=1}^{n} \phi^l_{(i_1,\dots,i_r|j_{r+1},\dots,j_k)} \Lambda_l^{(i_1,\dots,i_r|j_{r+1},\dots,j_k)},$$

where the $\phi^l_{(i_1,\dots,i_r|j_{r+1},\dots,j_k)} \in \mathbb{K}$ are arbitrary scalars in the ground field. For more brevity and without much abuse of notation, let us denote $\phi^l_{(i|j)_{r,k}}$, $\Lambda_l^{(i|j)_{r,k}}$ and $(\mathsf{e}_i, \mathsf{o}_j)_{r,k}$ instead of $\phi^l_{(i_1,\dots,i_r|j_{r+1},\dots,j_k)}$, $\Lambda_l^{(i_1,\dots,i_r|j_{r+1},\dots,j_k)}$ and $(\mathsf{e}_{i_1}, \dots, \mathsf{e}_{i_r}, \mathsf{o}_{j_{r+1}}, \dots, \mathsf{o}_{j_k})$, respectively. Thus, with these abbreviated notations, the above expansion of a general k-cochain reads:

$$\Phi = \sum_r \sum_{i_1 < \cdots < i_r} \sum_{j_{r+1} \leqslant \cdots \leqslant j_k} \sum_l \phi^l_{(i|j)_{r,k}} \Lambda_l^{(i|j)_{r,k}}. \tag{4}$$

In the important (special) case of standard Lie algebras $\mathfrak{g} \subset \mathfrak{h} = V$ represented by means of bases:

$$\mathfrak{g} = \mathbb{K}\,\mathsf{e}_1 \oplus \cdots \oplus \mathbb{K}\,\mathsf{e}_m \quad \text{and} \quad \mathfrak{h} = \mathbb{K}\,\mathsf{f}_1 \oplus \cdots \oplus \mathbb{K}\,\mathsf{f}_n,$$

odd elements are plainly absent, whence the expression of a general k-cochain reduces to:

$$\Phi = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m} \sum_{l=1}^{n} \phi^l_{i_1,\dots,i_k} \Lambda_l^{i_1,\dots,i_k},$$

where the basic k-cochains $\Lambda_l^{i_1,\dots,i_k}$ also write as follows in terms of the dual e_i^*:

$$\Lambda_l^{i_1,\dots,i_k} = \mathsf{e}_{i_1}^* \wedge \cdots \wedge \mathsf{e}_{i_k}^* \otimes \mathsf{f}_l. \tag{5}$$

Now, in order to compute the cocycle subspace $\mathscr{Z}^k \subset \mathscr{C}^k$, one proceeds by applying the fundamental formula (1) to know what value $\partial^k \Phi$ has on each $(k+1)$-tuple $(\mathsf{e}_i, \mathsf{o}_j)_{s,k+1}$, for all $s = 0, \dots, k+1$, for all $1 \leqslant i_1 < \cdots < i_s \leqslant p$, for all $p+1 \leqslant j_{s+1} \leqslant \cdots \leqslant j_{k+1} \leqslant m$, and afterwards, by just equating to zero each such expression $(\partial^k \Phi)\big((\mathsf{e}_i, \mathsf{o}_j)_{s,k+1}\big)$, a task which is of course left to a computer. With more precisions, because each such $(\partial^k \Phi)\big((\mathsf{e}_i, \mathsf{o}_j)_{s,k+1}\big)$ belongs to the n-dimensional $\mathbb{K}$-vector space V, one in fact gets n scalar equations in this way. After all, this gives in sum a finite number of homogeneous equations that are all linear with respect to the unknown coefficients $\phi^l_{(i|j)_{r,k}}$. Then by computer-solving the obtained linear system which we shall denote by:

$$\mathsf{Syst}_\phi\big(\mathscr{Z}^k\big),$$

one completely identifies those coefficients $\phi^l_{(i|j)_{r,k}}$ which make up cocycles $\Phi = \sum \phi^l_{(i|j)_{r,k}} \Lambda_l^{(i|j)_{r,k}}$ which belong to $\mathscr{Z}^k$. The first step ends so.

3.2. **Computation of $\mathscr{B}^{\mathbf{k}}(\mathfrak{g}, \mathbf{V})$.** This second step is rather similar to the first one, though less direct, for it requires the use of elimination ideals (Proposition 2.4). Indeed using once more the general representation (4) with k replaced by $k-1$, a general $(k-1)$-cochain writes quite similarly under the form:

$$\Psi = \sum_{r=0}^{k-1} \sum_{1\leqslant i_1<\cdots<i_r\leqslant p} \sum_{p+1\leqslant j_{r+1}\leqslant\cdots\leqslant j_{k-1}\leqslant m} \sum_{l=1}^{n} \psi_{(i|j)_{r,k-1}}^{l} \, \Lambda_l^{(i|j)_{r,k-1}}, \tag{6}$$

where the $\psi_{(i|j)_{r,k-1}}^{l} \in \mathbb{K}$ are arbitrary scalars in the ground field. By definition, the elements of $\mathscr{B}^k$, namely the coboundaries, are k-cochains of the form $\partial^{k-1}\Psi$, for such a Ψ. With more precision, $\mathscr{B}^k$ is the space of k-cochains Φ as in (4) that are of the form $\Phi = \partial^{k-1}\Psi$, for some $(k-1)$-cochains Ψ as in (6). Consequently, applying once again the fundamental formula (1), we have to compute the value of $\partial^{k-1}\Psi$ on each of the k-tuples $(\mathsf{e}_i, \mathsf{o}_j)_{r,k}$ belonging to $\mathfrak{g}^k$ and then to equate them to the value of Φ on these k-tuples, where we recall that:

$$\Phi\big((\mathsf{e}_i, \mathsf{o}_j)_{r,k}\big) = \Phi(\mathsf{e}_{i_1}, \ldots, \mathsf{e}_{i_r}, \mathsf{o}_{j_{r+1}}, \ldots, \mathsf{o}_{j_k}) = \sum_{l=1}^{n} \phi_{(i_1,\ldots,i_r|j_{r+1},\ldots,j_k)}^{l} \, v_l.$$

But looking at (1), and without performing explicit computations (left to a computer in specific examples), one easily convinces oneself that there are certain *linear* forms $\mathsf{L}_{i,j,r,k}$ in the coefficients $\psi_{(i'|j')_{r',k-1}}^{l'}$ of Ψ such that:

$$(\partial^{k-1}\Psi)\big((\mathsf{e}_i, \mathsf{o}_j)_{r,k}\big) = \sum_{l=1}^{n} \mathsf{L}_{i,j,r,k}\big(\big\{\psi_{(i'|j')_{r',k-1}}^{l'}\big\}\big) \, v_l.$$

Hence for any i, j, r, k, by equating the coefficients of the v_l, $l = 1, \ldots, n$, in both sides of the equalities:

$$\partial^{k-1}\Psi\big((\mathsf{e}_i, \mathsf{o}_j)_{r,k}\big) = \Phi\big((\mathsf{e}_i, \mathsf{o}_j)_{r,k}\big),$$

it therefore follows that a k-cochain $\Phi = \partial^{k-1}\Psi$ is a k-coboundary if and only if all its coefficients $\phi_{(i|j)_{r,k}}^{l}$ are of the form:

$$\phi_{(i|j)_{r,k}}^{l} = \mathsf{L}_{i,j,r,k}\Big(\big\{\psi_{(i'|j')_{r',k-1}}^{l'}\big\}\Big),$$

for *some* $(k-1)$-cochain Ψ having coefficients $\psi_{(i'|j')_{r',k-1}}^{l'}$. The task of writing explicitly the right-hand sides being left to a computer, we obtain in this way a finite number of linear equations. Lastly, we can use Gröbner bases to *eliminate* all the variables $\psi_{(i'|j')_{r',k-1}}^{l'}$ in these linear equations (*cf.* Proposition 2.4), which provides at the end a collection of linear equations (automatically organized as a reduced Gröbner basis) involving only the variables $\phi_{(i|j)_{r,k}}^{l}$. If we denote this new system by:

$$\mathsf{Syst}_\phi\big(\mathscr{B}^k\big),$$

the fact that one always has $\mathscr{B}^k \subset \mathscr{Z}^k$ entails that any solution of $\mathsf{Syst}_\phi\big(\mathscr{B}^k\big)$ is necessarily a solution of $\mathsf{Syst}_\phi\big(\mathscr{Z}^k\big)$. However as usual in linear algebra, this does not mean that the (finite) collection of equations for $\mathsf{Syst}_\phi\big(\mathscr{Z}^k\big)$ is *included*, as a set, in the (finite) collection of equations for $\mathsf{Syst}_\phi\big(\mathscr{B}^k\big)$: one in general needs to make linear combinations until this becomes true.

3.3. **Computation of $\mathbf{H^k}(\mathfrak{g}, \mathbf{V})$.** Now we are ready to start the third, main step, namely the computation of the k-th cohomological space $H^k = \mathscr{Z}^k \big/ \mathscr{B}^k$. (Of course, any technique which decreases the complexity of this last step simultaneously increases the speediness of computations.) The two systems $\mathsf{Syst}_\phi(\mathscr{Z}^k)$ and $\mathsf{Syst}_\phi(\mathscr{B}^k)$ of linear equations in the unknown variables $\phi^l_{(i|j)_{r,k}}$ identify exactly all the elements of $\mathscr{Z}^k$ and $\mathscr{B}^k$, respectively. Therefore, every nonzero element of the quotient $\mathbb{K}$-vector space:

$$H^k = \mathscr{Z}^k \big/ \mathscr{B}^k$$

is of the form:

$$\Phi + \mathscr{B}^k,$$

where the coefficients $\phi_l^{(i|j)_{r,k}}$ of the k-cochain $\Phi = \sum \phi^l_{(i|j)_{r,k}} \Lambda_l^{(i|j)_{r,k}}$ satisfy all the equations in $\mathsf{Syst}_\phi(\mathscr{Z}^k)$ and do not satisfy at least one of the equations in $\mathsf{Syst}_\phi(\mathscr{B}^k)$.

3.4. **Finding a basis for a quotient $\mathbb{K}$-vector space.** Temporarily, let us set aside our cohomological objective and let us present some results in the theory of Gröbner basis that are useful to the purpose of finding representatives of the quotient V/W of any two $\mathbb{K}$-vector subspaces $W \subset V \subset E$ sitting inside a certain (large) ambient $\mathbb{K}$-vector space E.

In a first moment, given a vector subspace $F \subset E$ of some $\mathbb{K}$-vector space E which is represented as the zero-set of some linear forms — as for instance $\mathscr{Z}^k \subset \mathscr{C}^k$ which is represented by $\mathsf{Syst}_\phi\big(\mathscr{Z}^k\big)$ —, by allowing fully the use of Gröbner bases, we want to find an explicit set of vectors $\mathsf{f}_1, \dots, \mathsf{f}_{\dim F} \in E$ which make up a basis for F. Then in a second moment and still employing Gröbner bases, given instead two $\mathbb{K}$-vector subspaces $W \subset V \subset E$ of dimensions $p := \dim_{\mathbb{K}} V$ and $q := \dim_{\mathbb{K}} W$ which are both represented as zero-sets of some linear forms — as for instance $\mathscr{B}^k \subset \mathscr{Z}^k \subset \mathscr{C}^k$ which are represented by $\mathsf{Syst}_\phi\big(\mathscr{B}^k\big)$ and by $\mathsf{Syst}_\phi\big(\mathscr{Z}^k\big)$ —, we will show how to find explicitly $p - q$ linearly independent vectors $\mathsf{v}_1, \dots, \mathsf{v}_{p-q} \in V$ such that the cosets:

$$\mathsf{v}_1 + W, \ \dots, \ \mathsf{v}_{p-q} + W$$

make up a basis for the quotient vector space V/W (following notation from [18], pp. 347–348).

Thus, let E be a $\mathbb{K}$-vector space of dimension $n \geqslant 1$, let $\{\mathsf{e}_1, \dots, \mathsf{e}_n\}$ be a basis of E and let $(x_1, \dots, x_n) \in \mathbb{K}^n$ be the associated coordinates in terms of which any vector $\mathsf{e} \in E \simeq \mathbb{K}^n$ represents uniquely as:

$$\mathsf{e} = x_1 \, \mathsf{e}_1 + \cdots + x_n \, \mathsf{e}_n.$$

By convention, the variable names x_i will be reserved to write down Cartesian equations of vector subspaces, and we will also need some other auxiliary variables $(y_1, \dots, y_n)$. Often, x and y will abbreviate $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$. With slight abuse, polynomials in $\mathbb{K}[x]$ will sometimes be written $f(x)$ — with 'argument' x —, in order to see without ambiguity the indeterminate which will be either x or y, and this functional notation is justified by the fact that to any polynomial $P = a_0 + a_1 \, t + \cdots + a_n \, t^n \in \mathbb{K}[t]$ is associated the map $\mathbb{K} \ni t \longmapsto a_0 + a_1 \, t + \cdots + a_n \, t^n \in \mathbb{K}$.

To begin with, consider the circumstance where a given vector subspace $F \subset E \simeq \mathbb{K}^n$ is represented as generated by μ vectors $\mathsf{f}_1, \dots, \mathsf{f}_\mu \in F$ that are not necessarily linearly independent. Each such vector decomposes according to the basis:

$$\mathsf{f}_1 = f_{11}\, \mathsf{e}_1 + \cdots + f_{1n}\, \mathsf{e}_n, \;\; \dots\dots, \;\; \mathsf{f}_\mu = f_{\mu 1}\, \mathsf{e}_1 + \cdots + f_{\mu n}\, \mathsf{e}_n,$$

for some scalars $f_{\lambda i} \in \mathbb{K}$, and using the auxiliary variables $(y_1, \dots, y_n)$, we associate to them the following μ linear forms:

$$f_1(y) := f_{11}\, y_1 + \cdots + f_{1n}\, y_n, \;\; \dots\dots, \;\; f_\mu(y) := f_{\mu 1}\, y_1 + \cdots + f_{\mu n}\, y_n,$$

which we simply view as (degree 1) *polynomials* belonging to $\mathbb{K}[y_1, \dots, y_n]$. The proofs of the three statements below, including the following preliminary proposition, will be postponed to the end of the present section.

Proposition 3.1. *Fix a lexicographic ordering $\prec$ on monomials of the ring $\mathbb{K}[y_1, \dots, y_n]$. With $F = \mathrm{Vect}_{\mathbb{K}}(\mathsf{f}_1, \dots, \mathsf{f}_\mu)$ as above, and with the associated linear forms $f_1(y), \dots, f_\mu(y)$, if $\mathtt{G} := \{g_1(y), \dots, g_m(y)\}$ is* the *reduced Gröbner basis of the ideal:*

$$\big\langle\, f_1(y), \dots, f_\mu(y) \,\big\rangle$$

in $\mathbb{K}[y_1, \dots, y_n]$ with respect to $\prec$, then:

- **(i)** $\dim_{\mathbb{K}} F = m =$ *precisely the cardinal of* $\mathtt{G}$;
- **(ii)** *all $g_j(y)$, $j = 1, \dots, m$, are linear forms, namely:*

$$g_j(y) = g_{j1}\, y_1 + \cdots + g_{jn}\, y_n$$

for some scalars $g_{ji} \in \mathbb{K}$, and furthermore, the m vectors:

$$\mathsf{g}_1 := g_{j1}\, \mathsf{e}_1 + \cdots + g_{jn}\, \mathsf{e}_n, \;\; \dots\dots, \;\; \mathsf{g}_m := g_{m1}\, \mathsf{e}_1 + \cdots + g_{mn}\, \mathsf{e}_n$$

constitute a basis for F as a vector space;

- **(iii)** *an arbitrary vector $\mathsf{h} = h_1\, \mathsf{e}_1 + \cdots + h_n\, \mathsf{e}_n \in E$, with coordinates $h_i \in \mathbb{K}$, belongs to F* if and only if *the normal form of the associated $h(y) := h_1\, y_1 + \cdots + h_n\, y_n$ with respect to the reduced Gröbner basis $\mathtt{G}$ is zero:*

$$0 = \mathrm{NF}_{\mathtt{G}}(h).$$

However, as we said, the $\mathbb{K}$-vector subspace $F \subset E$ we want to consider for applications to (super) Lie algebra cohomologies, namely $\mathscr{Z}^k \subset \mathscr{C}^k$ (or also $\mathscr{B}^k \subset \mathscr{C}^k$) should be thought of as being represented as the zero-set of some (Cartesian) linear equations. The appropriate statement will better be brought to light by means of a simple illustration.

Example 3.2. Consider the system of three (Cartesian) linear equations:

$$\begin{cases} f_1(x) := x_1 - x_4 + x_5 = 0, \\ f_2(x) := 2\, x_1 + x_2 + x_4 = 0, \\ f_3(x) := -x_3 + 2\, x_4 + x_5 = 0, \end{cases}$$

in the vector space $E = \mathbb{K}^5$ with coordinates $(x_1, x_2, x_3, x_4, x_5)$ which represents a certain vector subspace $F \subset E$. Transforming (either by hand or with a computer) the ideal $\big\langle\, f_1(x),\, f_2(x),\, f_3(x) \,\big\rangle$ to the reduced Gröbner basis with respect to the lexicographic ordering $x_5 \prec x_4 \prec x_3 \prec x_2 \prec x_1$, one gets that

$F \subset E$ is equivalently defined as the set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{K}$ satisfying: $0 = g_1(x) = g_2(x) = g_3(x)$, where:

$$g_1(x) := x_1 - x_4 + x_5, \qquad g_2(x) := x_2 + 3\,x_4 - 2\,x_5, \qquad g_3(x) := x_3 - 2\,x_4 - x_5,$$

and where $\mathtt{G} := \{g_1(x), g_2(x), g_3(x)\}$ is the reduced Gröbner basis in question. Thus, x_4 and x_5, are *horizontal parameters* for F, x_1, x_2, x_3 are functions of (x_4, x_5), and F is a *graphed* $5-3 = 2$-dimensional subspace of the 5-dimensional vector space $E = \mathbb{K}^5$.

Next, choosing firstly $(x_4, x_5) = (1, 0)$ and secondly $(x_4, x_5) = (0, 1)$, one sees that F is generated by the two column vectors $(1, -3, 2, 1, 0)^{\mathsf{t}}$ and $(-1, 2, 1, 0, 1)^{\mathsf{t}}$. To these two vectors, one then associates the following set of two linear forms:

$$\Big\{y_1 - 3y_2 + 2y_3 + y_4,\ -y_1 + 2y_2 + y_3 + y_5\Big\},$$

in some five auxiliary variables $y_1, y_2, y_3, y_4, y_5 \in \mathbb{K}$. On the other hand, granted that computing a normal form with respect to $\mathtt{G}$ just means replacing x_1 by $x_4 - x_5$, x_2 by $-3\,x_4 + 2\,x_5$ and x_3 by $2\,x_4 + x_5$, and considering the auxiliary bilinear form $\sum_{i=1}^5 x_i\,y_i$, we see that:

$$\mathrm{NF}_{\mathtt{G}}\Big(\textstyle\sum_{i=1}^5 x_i\,y_i\Big) = (x_4 - x_5)\,y_1 + (-3\,x_4 + 2\,x_5)\,y_2 + (2\,x_4 + x_5)\,y_3 + x_4\,y_4 + x_5\,y_5.$$

Reorganizing, we easily find the coefficients of the parameters x_4 and x_5 in this expression:

$$\begin{aligned} \boxed{x_4}: &\qquad y_1 - 3\,y_2 + 2\,y_3 + y_4 \\ \boxed{x_5}: &\qquad -\,y_1 + 2\,y_2 + y_3 + y_5, \end{aligned}$$

and interestingly enough, these two coefficients coincide with the above two linear forms in the auxiliary variables y_i. This is a quite general fact, whose proof is also postponed to the end of the present section.

Proposition 3.3. *Let $F \subset E \simeq \mathbb{K}^n$ be a $\mathbb{K}$-vector subspace which is represented by means of Cartesian linear equations:*

$$F = \big\{\text{vectors } x_1\,\mathsf{e}_1 + \cdots + x_n\,\mathsf{e}_n \text{ s.t. } 0 = f_1(x) = \cdots = f_\mu(x)\big\},$$

for a certain collection of $\mu \geqslant 1$ linear forms $f_\lambda(x)$. Let $\mathtt{G}$ be the reduced Gröbner basis of the ideal $\big\langle\, f_1(x), \ldots, f_\mu(x)\,\big\rangle$ with respect to some fixed lexicographic *ordering. Given n new auxiliary indeterminates $y_1, \ldots, y_n$, let:*

$$h_y(x) := \mathrm{NF}_{\mathtt{G}}\big(x_1 y_1 + \cdots + x_n y_n\big) \in \mathbb{K}[x_1, \ldots, x_n]$$

be the normal form, with respect to $\mathtt{G}$, of the bilinear form $\sum_{i=1}^n x_i\,y_i$. Then the following four assertions hold true:

- **(i)** *$h_y(x)$ is linear in $(x_1, \ldots, x_n)$;*
- **(ii)** *$h_y(x)$ involves exactly* $\dim F =: m$ *variables x_i:*

$$h_y(x) = x_{i_1}\,h_1(y) + \cdots + x_{i_m}\,h_m(y),$$

  for some $1 \leqslant i_1 < \cdots < i_m \leqslant n$;
- **(iii)** *all the appearing coefficients $h_j(y)$ of $h_y(x)$ are linear forms in the variables $(y_1, \ldots, y_n)$;*

(iv) *if one expands them:*

$$h_j(y) = h_{j1}\, y_1 + \cdots + h_{jn}\, y_n \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,m)}$$

in terms of some scalars $h_{ji} \in \mathbb{K}$, then the m associated vectors:

$$\mathsf{h}_1 := h_{11}\, \mathsf{e}_1 + \cdots + h_{1n}\, \mathsf{e}_n, \ \ \ldots\ldots, \ \ \mathsf{h}_m := h_{m1}\, \mathsf{e}_1 + \cdots + h_{mn}\, \mathsf{e}_n$$

make up a basis for F.

The last data $\mathsf{h}_1, \ldots, \mathsf{h}_m$ are exactly what we wanted: an explicit basis for the $\mathbb{K}$-vector subspace $F \subset E$ which was represented by linear equations.

We can now come back to our initial goal. Let $E \simeq \mathbb{K}^n$ be an ambient n-dimensional $\mathbb{K}$-vector space as above, fix coordinates $(x_1, \ldots, x_n)$ on E and fix some *lexicographic* ordering on monomials of $\mathbb{K}[x_1, \ldots, x_n]$. Let $W \subset E$ and $V \subset E$ be two $\mathbb{K}$-vector subspaces which are both represented by means of Cartesian linear equations:

$$\begin{aligned} W &= \big\{\text{vectors } x_1\, \mathsf{e}_1 + \cdots + x_n\, \mathsf{e}_n \text{ s.t. } 0 = g_1(x) = \cdots = g_\nu(x)\big\}, \\ V &= \big\{\text{vectors } x_1\, \mathsf{e}_1 + \cdots + x_n\, \mathsf{e}_n \text{ s.t. } 0 = f_1(x) = \cdots = f_\mu(x)\big\}, \end{aligned}$$

for certain two collections of linear forms $g_1(x), \ldots, g_\nu(x)$ and $f_1(x), \ldots, f_\mu(x)$, with the further assumption that $W \subset V$. For our cohomological objective, the initial data are precisely presented under such form: $\mathscr{B}^k \subset \mathscr{C}^k$ and $\mathscr{Z}^k \subset \mathscr{C}^k$ are the zero-sets of $\mathsf{Syst}_\phi\big(\mathscr{B}^k\big)$ and of $\mathsf{Syst}_\phi\big(\mathscr{B}^k\big)$, respectively, with $\mathscr{B}^k \subset \mathscr{Z}^k$, of course. It goes without saying that Proposition 3.3 provides two explicit bases for W and V, namely:

$$W = \operatorname{Span}_{\mathbb{K}}\big(\mathsf{w}_1, \ldots, \mathsf{w}_q\big) \qquad \text{and} \qquad V = \operatorname{Span}_{\mathbb{K}}\big(\mathsf{v}_1, \ldots, \mathsf{v}_p\big),$$

where $q := \dim_{\mathbb{K}} W$ and $p := \dim_{\mathbb{K}} V$. The following theorem then realizes the goal of finding a basis for V/W as a $\mathbb{K}$-vector space.

Theorem 3.1. *Let E be an n-dimensional $\mathbb{K}$-vector space equipped with a basis $\{\mathsf{e}_1, \ldots, \mathsf{e}_n\}$, let $V \subset E$ and $W \subset E$ be two $\mathbb{K}$-vector subspaces having dimensions $p := \dim_{\mathbb{K}} V$ and $q := \dim_{\mathbb{K}} W$ that are both represented:*

$$V = \operatorname{Span}_{\mathbb{K}}\big(\mathsf{v}_1, \ldots, \mathsf{v}_p\big) \qquad \text{and} \qquad W = \operatorname{Span}_{\mathbb{K}}\big(\mathsf{w}_1, \ldots, \mathsf{w}_q\big),$$

as the span of some basis vectors:

$$\underset{(i\,=\,1\,\cdots\,p)}{\mathsf{v}_i = v_{i1}\, \mathsf{e}_1 + \cdots + v_{in}\, \mathsf{e}_n} \qquad \text{and} \qquad \underset{(j\,=\,1\,\cdots\,q)}{\mathsf{w}_j = w_{j1}\, \mathsf{e}_1 + \cdots + w_{jn}\, \mathsf{e}_n}$$

which are explicitly given in terms of their coordinates $v_{ik} \in \mathbb{K}$ and $w_{jk} \in \mathbb{K}$. Suppose that $W \subset V$, whence $q \leqslant p$, and associate to these two bases the following two collections of linear forms:

$$f_i(y) := v_{i1}\, y_1 + \cdots + v_{in}\, y_n \qquad \text{and} \qquad g_j(y) := w_{j1}\, y_1 + \cdots + w_{jn}\, y_n$$

in some auxiliary $\mathbb{K}[y_1, \ldots, y_n]$. Lastly, let:

$$\mathsf{B}_V := \big\{\overline{f}_1(y), \ldots, \overline{f}_p(y)\big\} \qquad \text{and} \qquad \mathsf{B}_W := \big\{\overline{g}_1(y), \ldots, \overline{g}_q(y)\big\}$$

be the two reduced Gröbner bases of the two ideals $\big\langle\, f_1(y),\ldots,f_p(y)\,\big\rangle$ *and* $\big\langle\, g_1(y),\ldots,g_q(y)\,\big\rangle$ *with respect to some fixed* lexicographic *ordering* $\prec$ *on the monomials of* $\mathbb{K}[y_1,\ldots,y_n]$*. Then the reduced Gröbner basis* $\mathtt{B}_{V/W}$ *of the ideal:*

$$\Big\langle\, \mathrm{NF}_{\mathtt{B}_W}\big(\,\overline{f}\,\big)\,:\ \overline{f}\in\mathtt{B}_V\,\Big\rangle$$

generated by the normal forms with respect to $\mathtt{B}_W$ *of all elements of* $\mathtt{B}_V$*, is of cardinal equal to* $p-q=\dim V-\dim W$*, and furthermore, if:*

$$\overline{h}_l(y)=h_{l1}\,y_1+\cdots+h_{ln}\,y_n \qquad\qquad {\scriptstyle (l\,=\,1\,\cdots\,p\,-\,q)}$$

are its elements, the $p-q$ *associated vectors:*

$$\mathsf{h}_l:=h_{l1}\,\mathsf{e}_1+\cdots+h_{ln}\,\mathsf{e}_n \qquad\qquad {\scriptstyle (l\,=\,1\,\cdots\,p\,-\,q)}$$

belong to V *and the cosets* h_l+W *make up a basis for* V/W*.*

Computer tests (*cf.* examples below) show that, compared with standard linear algebra methods, the use of Gröbner bases improves speed and efficiency, especially because the computations underlying Proposition 3.3 and Theorem 3.1 can be achieved within a polynomial ring, without the need of several transformations between polynomials and vectors; indeed, from the two collections of Cartesian linear equations $\mathsf{Syst}_\phi\big(\mathscr{Z}^k\big)$ and $\mathsf{Syst}_\phi\big(\mathscr{B}^k\big)$, Proposition 3.3 extracts two collections of polynomials in some auxiliary variables $v_l^{(i|j)_{r,k}}$ to which one can directly apply Theorem 3.1 in order to find a basis for the sought cohomology space $H^k=\mathscr{Z}^k\big/\mathscr{B}^k$, *see* also the description of the algorithm in the next section. One further reason why the use of the standard Gauss-Jordan elimination through pivoting is less quick when applied to the examples we know from differential geometry, is probably that the associated matrices are large, though plenty of 0's, hence the computer must make many operations with large lines or rows. But when one translates the cohomology computation problem in terms of degree 1 polynomials as above, the 0's disappear, just existing terms are taken account of.

Proof of Proposition 3.1. We begin by making a preliminary observation. According to the process of producing any Gröbner basis, each element $g_j(y)$ of $\mathtt{G}$ is obtained by subjecting all pairs $\{f_{\lambda_1}(y),f_{\lambda_2}(y)\}$ to an S-polynomial elimination of leading terms, by performing division (Theorem 2.1) and by repeating the process until stabilization, whence one easily convinces oneself that *only linear forms*, namely degree one polynomials having no constant term, can come up at each stage. At the end, every $g_j(y)$ is therefore a linear form. Of course, the ideal is the same:

$$\langle\, f_1(y),\ldots,f_\mu(y)\,\rangle=\langle\, g_1(y),\ldots,g_m(y)\,\rangle\,.$$

Thus, because all considered polynomials are linear forms, there necessarily exist some scalars $c_{j\lambda}\in\mathbb{K}$ such that $g_j(y)=\sum_{\lambda=1}^{\mu}c_{j\lambda}\,f_\lambda(y)$ for all $j=1,\ldots,m$, and in the other direction also, there necessarily exist some scalars $d_{\lambda j}\in\mathbb{K}$ such that $f_\lambda(y)=\sum_{j=1}^{m}d_{\lambda j}\,g_j(y)$ for all $\lambda=1,\ldots,\mu$. It follows that the vector subspace $F_{\mathtt{G}}$ associated to the g_j by **(ii)** is contained in the original vector subspace $F\subset E$ to which the $f_\lambda(y)$ were associated, and also in the other direction that $F\subset F_{\mathtt{G}}$. Consequently, we have $F=F_{\mathtt{G}}$.

To finish with **(i)** and **(ii)**, it remains to prove the linear independency of the vectors $\mathsf{g}_1, \dots, \mathsf{g}_m$ associated to $g_1(y), \dots, g_m(y)$. Suppose by contradiction that $0 = c_1\, \mathsf{g}_1 + \cdots + c_m\, \mathsf{g}_m$ for some $c_i \in \mathbb{K}$ that are not all zero. It immediately follows that $c_1\, g_1(y) + \cdots + c_m\, g_m(y) \equiv 0$. Consequently there exist at least two different integers $j_1 \neq j_2$ such that $\mathrm{LM}(g_{j_1}) = \mathrm{LM}(g_{j_2})$, contrarily to the assumption that the chosen $\mathtt{G}$ was a *reduced* Gröbner basis. In sum:

$$m = \mathrm{Card}\, \mathtt{G} = \dim_{\mathbb{K}} F.$$

Lastly, we check **(iii)**. Of course, a vector h belongs to $F = F_{\mathtt{G}}$ if and only if thre exist scalars $c_i \in \mathbb{K}$ such that $\mathsf{h} = c_1\, \mathsf{g}_1 + \cdots + c_m\, \mathsf{g}_m$. Equivalently, the associated polynomial (linear form) $h(y) = c_1\, g_1(y) + \cdots + c_m\, g_m(y)$ belongs to the ideal generated by the Gröbner basis $\mathtt{G}$. But this is so if and only if the normal form $\mathrm{NF}_{\mathtt{G}}(h)$ of $h(y)$ with respect to $\mathtt{G}$ is zero. □

Proof of Proposition 3.3. We already saw, in the beginning of the proof of the preceding proposition, that all elements of $\mathtt{G}$ are linear forms and that any division by $\mathtt{G}$ preserves linearity in $\mathbb{K}[x_1, \dots, x_n]$. Since $\sum_{i=1}^n x_i\, y_i$ is linear in the x_i, its normal form $h_y(x)$ with respect to $\mathtt{G}$ is also linear, which is **(i)**.

Next, let $\underline{m}$ denote the cardinal of the reduced Gröbner basis $\mathtt{G}$ of the ideal $\big\langle f_1(x), \dots, f_\mu(x) \big\rangle$, and denote its elements by $g_1(x), \dots, g_{\underline{m}}(x)$. Since $\mathtt{G}$ is reduced, for all $l = 1, \dots, \underline{m}$, the leading terms of $g_l(x)$ are monic, of degree one of course, and distinct, say:

$$\mathrm{LT}(g_1) = x_{i_1}, \dots, \mathrm{LT}(g_{\underline{m}}) = x_{i_{\underline{m}}} \qquad \text{for some } 1 \leqslant i_1 < \cdots < i_{\underline{m}} \leqslant n.$$

Again because $\mathtt{G}$ is reduced, each g_l does not contain any $x_{i_1}, \dots, x_{i_{\underline{m}}}$, aside from its leading term x_{i_l}. After relabelling the x_i if necessary, we can (and we shall) assume that $i_1 = n - \underline{m} + 1, \dots, i_{\underline{m}} = n$. Then the g_l write under a graphed form:

$$g_l\big(x_1, \dots, x_{n-\underline{m}},\, x_{n-\underline{m}+1}, \dots, x_n\big) = x_l - g_l'(x_1, \dots, x_{n-\underline{m}}) \qquad {\scriptstyle (l = n - \underline{m} + 1, \dots, n)},$$

for some linear forms g_l' in only the $n - \underline{m}$ first variables $x_1, \dots, x_{n-\underline{m}}$. But then, since the vector subspace $F \subset E$ is as well represented by the corresponding $\underline{m}$ Cartesian linear equations $0 = x_l - g_l'(x_1, \dots, x_{\underline{m}})$, for $l = n - \underline{m} + 1, \dots, n$, it goes without saying that, in the notation of the proposition:

$$m := \dim_{\mathbb{K}} F = n - \underline{m},$$

so that we can replace $\underline{m}$ by $n - m$ everywhere. Furthermore, if we expand:

$$g_l'(x_1, \dots, x_m) = \sum_{j=1}^m g_{lj}'\, x_j \qquad {\scriptstyle (l = m + 1 \cdots n)}$$

with some scalars $g_{lj}' \in \mathbb{K}$, it is clear that a certain basis for F which is naturally associated to the Cartesian linear equations in question just consists of the m vectors obtained by setting one x_j equal to 1 and the others equal to 0, for any choice of $j = 1, \dots, m$, which yields the m vectors:

$$\mathsf{e}_j + \sum_{l=m+1}^n g_{lj}'\, \mathsf{e}_l \qquad {\scriptstyle (j = 1 \cdots m)}. \tag{7}$$

On the other hand, the reduction of the auxiliary bilinear form $\sum_{i=1}^n x_i\,y_i$ to normal form with respect to $\mathtt{G}$ then just means replacing x_l by $g_l'(x_1,\dots,x_m)$, for $l=m+1,\dots,n$, so that:

$$\begin{aligned} h_y(x) = \mathrm{NF}_{\mathtt{G}}\Big(\textstyle\sum_{i=1}^n x_i y_i\Big) &= \sum_{j=1}^n x_j\,y_j + \sum_{l=m+1}^n g_l'(x_1,\dots,x_m)\,y_l \\ &= \sum_{j=1}^m x_j\,y_j + \sum_{l=m+1}^n \sum_{j=1}^m g_{lj}'\,x_j\,y_l \\ &= \sum_{j=1}^m x_j \underbrace{\Bigg(y_j + \sum_{l=m+1}^n g_{lj}'\,y_l\Bigg)}_{=:h_j(y)}, \end{aligned}$$

and from this last expression, one realizes that the m vectors:

$$\mathsf{h}_j = \mathsf{e}_j + \sum_{l=m+1}^n g_{lj}'\,\mathsf{e}_l \qquad\qquad (j=1\cdots m)$$

associated to the obtained coefficients $h_j(y)$ of $h_y(x)$ with respect to $x_1,\dots,x_m$ do indeed coincide with the $m=\dim F$ vectors (7) which were seen to constitute a basis for F a moment ago. The simultaneous proof of properties **(ii)**, **(iii)**, **(iv)** is therefore complete. □

Proof of Theorem 3.1. After a permutation of both the $\overline{g}_j$ and the variables y_i, we can assume that the lexicographic ordering is just $y_n \prec \cdots \prec y_2 \prec y_1$ and that the q leading terms of the generators $\overline{g}_1(y),\dots,\overline{g}_q(y)$ of the Gröbner basis $\mathtt{B}_W$ are just $y_1,\dots,y_q$. Since $\mathtt{B}_W$ is reduced, its q elements necessarily write under a graphed, linear form:

$$\mathtt{B}_W = \Big\{\underbrace{y_j - \textstyle\sum_{i=q+1}^{i=n} b_{j,i}\,y_i}_{\overline{g}_j(y)}\Big\}_{1\leqslant j\leqslant q}$$

for some scalars $b_{\bullet,\bullet}\in\mathbb{K}$. Similarly, the p elements $\overline{f}_1(y),\dots,\overline{f}_p(y)$ of the Gröbner basis $\mathtt{B}_V$ must also be of a certain graphed, linear form. Let $q'\leqslant q$ be the number of leading terms of elements of $\mathtt{B}_V$ that are equal to one leading term y_j with $1\leqslant j\leqslant q$ appearing in the members of $\mathtt{B}_W$. Possibly after an independent renumbering of both $y_1,\dots,y_q$ and $y_{q+1},\dots,y_n$, it follows that there is a decomposition of the y_i-variables into four groups of variables:

$$\big(\underline{y_1,\dots,y_{q'}},\,y_{q'+1},\dots,y_q,\,\underline{y_{q+1},\dots,y_{p+q-q'}},\,y_{p+q-q'+1},\dots,y_n\big)$$

such that the $p=q'+(p-q')$ elements of $\mathtt{B}_V$ do precisely have those leading monomials that are underlined and do write under the following graphed form:

$$\begin{aligned} \mathtt{B}_V = \Big\{y_{j'} - \textstyle\sum_{i=q'+1}^{i=q} a_{j',i}\,y_i - \sum_{i=p+q-q'+1}^{i=n} a_{j',i}\,y_i\Big\}_{1\leqslant j'\leqslant q'} \bigcup \\ \bigcup \Big\{y_l - \textstyle\sum_{i=q'+1}^{i=q} a_{l,i}\,y_i - \sum_{i=p+q-q'+1}^{i=n} a_{l,i}\,y_i\Big\}_{q+1\leqslant l\leqslant p+q-q'} \end{aligned}$$

for some scalars $a_{\bullet,\bullet} \in \mathbb{K}$. However, all $a_{l,i}$ in the first sum of the second line must necessarily be equal to 0, because by assumption, we have:

$$y_l \prec y_{q'+1}, \dots, y_q \qquad \text{for all } \; q+1 \leqslant l \leqslant p+q-q',$$

whence if some $a_{l,i}$ would be nonzero, the number q' defined above would be larger. Thus, after simply erasing these $a_{l,i}$, it remains:

$$\begin{aligned}\mathtt{B}_V = \Big\{ y_{j'} - \textstyle\sum_{i=q'+1}^{i=q} a_{j',i}\, y_i - \sum_{i=p+q-q'+1}^{i=n} a_{j',i}\, y_i \Big\}_{1 \leqslant j' \leqslant q'} \bigcup \\ \bigcup \Big\{ y_l - \textstyle\sum_{i=p+q-q'+1}^{i=n} a_{l,i}\, y_i \Big\}_{q+1 \leqslant l \leqslant p+q-q'}.\end{aligned}$$

But now, we recall the assumption $W \subset V$ which reads in terms of ideals naturally as the constraint $\langle\, \mathtt{B}_W \,\rangle \subset \langle\, \mathtt{B}_V \,\rangle$. Since all existing polynomials are (degree-one) linear forms, each element $y_j - \sum_{i=q+1}^{i=n} b_{j,i}\, y_i$ of $\mathtt{B}_W$ for $j = q'+1, \dots, q$ must in particular be a certain linear combination of elements of $\mathtt{B}_V$ with scalar (degree-zero) coefficients. But all elements of $\mathtt{B}_V$ above are under a graphed form, with no such y_j with $j = q'+1, \dots, q$ appearing in either the $y_{j'}$ or in the y_l of $\mathtt{B}_V$, from what we deduce $q' = q$, whence immediately:

$$\mathtt{B}_V = \Big\{ y_j - \textstyle\sum_{i=p+1}^{i=n} a_{j,i}\, y_i \Big\}_{1 \leqslant j \leqslant q} \bigcup \Big\{ y_l - \sum_{i=p+1}^{i=n} a_{l,i}\, y_i \Big\}_{q+1 \leqslant l \leqslant p}.$$

Now that $q' = q$, the constraint $\langle\, \mathtt{B}_W \,\rangle \subset \langle\, \mathtt{B}_V \,\rangle$ means that, for every $j = 1, \dots, q$, there exist scalars $\lambda_{j,j_1} \in \mathbb{K}$ and $\mu_{j,l_1} \in \mathbb{K}$ such that one has:

$$\begin{aligned} y_j - \textstyle\sum_{i=q+1}^{i=n} b_{j,i}\, y_i \equiv \displaystyle\sum_{j_1=1}^{j_1=q} \lambda_{j,j_1} \Big(y_{j_1} - \textstyle\sum_{i=p+1}^{i=n} a_{j_1,i}\, y_i \Big) + \\ + \displaystyle\sum_{l_1=q+1}^{l_1=p} \mu_{j,l_1} \Big(y_{l_1} - \textstyle\sum_{i=p+1}^{i=n} a_{l_1,i}\, y_i \Big),\end{aligned}$$

identically in $\mathbb{K}[y_1, \dots, y_n]$. It necessarily follows that $\lambda_{j,j} = 1$ while $\lambda_{j,j_1} = 0$ for $j_1 \neq j$ and that:

$$-\, b_{j,i} = \mu_{j,i} \qquad \text{for } \; i = q+1, \dots, p.$$

After simplifying terms which cancel out, there remain the q equations:

$$-\textstyle\sum_{i=p+1}^{i=n} b_{j,i}\, y_i \equiv -\sum_{i=p+1}^{i=n} a_{j,i}\, y_i + \sum_{l_1=q+1}^{l_1=p} \sum_{i=p+1}^{i=n} b_{j,l_1}\, a_{l_1,i}\, y_i \qquad (j = 1 \cdots q),$$

holding identically in $\mathbb{K}[y_1, \dots, y_n]$, and this yields by identification of the coefficients of the y_i in both sides:

$$b_{j,i} = a_{j,i} - \textstyle\sum_{l_1=q+1}^{l_1=p} b_{j,l_1}\, a_{l_1,i} \qquad (j = 1 \cdots q\,;\; i = p+1 \dots n). \tag{8}$$

On the other hand, recalling that computing the normal form with respect to B_W just means replacing each y_j by $\sum_{i=q+1}^{i=n} b_{j,i}\, y_i$ for $j = 1, \dots, q$, we have:

$$\Big\langle \mathrm{NF}_{\mathsf{B}_W}(\overline{f}) \colon \overline{f} \in \mathsf{B}_V \Big\rangle = \Big\langle \Big\{ \sum_{i=q+1}^{i=n} b_{j,i}\, y_i - \sum_{i=p+1}^{i=n} a_{j,i}\, y_i \Big\}_{1 \leqslant j \leqslant q}, \Big\{ y_l - \sum_{i=p+1}^{i=n} a_{l,i}\, y_i \Big\}_{q+1 \leqslant l \leqslant p} \Big\rangle.$$

In the first family, we use the relation (8) obtained right above to replace the $b_{j,i}$ for $1 \leqslant j \leqslant q$ and for $p+1 \leqslant i \leqslant n$, which yields after a cancellation:

$$\Big\langle \mathrm{NF}_{\mathsf{B}_W}(\overline{f}) \colon \overline{f} \in \mathsf{B}_V \Big\rangle = \Big\langle \Big\{ \sum_{i=q+1}^{i=p} b_{j,i}\, y_i - \sum_{i=p+1}^{i=n} \sum_{l_1=q+1}^{l_1=p} b_{j,l_1}\, a_{l_1,i}\, y_i \Big\}_{1 \leqslant j \leqslant q}, \Big\{ y_l - \sum_{i=p+1}^{i=n} a_{l,i}\, y_i \Big\}_{q+1 \leqslant l \leqslant p} \Big\rangle.$$

But now, we observe that each element in the first family belongs in fact already to the ideal generated by the members of the second family, because the linear combination:

$$\sum_{l=q+1}^{l=p} b_{j,l} \Big(y_l - \sum_{i=p+1}^{i=n} a_{l,i}\, y_i \Big)$$

identifies, after change of indices, to the j-th element of the first family. In conclusion, the ideal:

$$\Big\langle \mathrm{NF}_{\mathsf{B}_W}(\overline{f}) \colon \overline{f} \in \mathsf{B}_V \Big\rangle = \Big\langle \Big\{ y_l - \sum_{i=p+1}^{i=n} a_{l,i}\, y_i \Big\}_{q+1 \leqslant l \leqslant p} \Big\rangle$$

is generated by exactly $p - q = \dim_{\mathbb{K}} V - \dim_{\mathbb{K}} W$ elements, with are *de facto* in reduced Gröbner basis form for the lexicographic ordering $\prec$, and the associated vectors:

$$\mathsf{h}_l = \mathsf{e}_l - \sum_{i=p+1}^{i=n} a_{l,i}\, \mathsf{e}_i \qquad (l = q+1 \cdots p)$$

belong to V by assumption (since vectors associated to elements of B_V belong to V) and are mutually linearly independent modulo W, as one can easily realize thanks to the fact that W is graphed over $\mathbb{K}\,\mathsf{e}_1 \oplus \cdots \oplus \mathbb{K}\,\mathsf{e}_q$. The proof of Theorem 3.1 is complete. □

4. Description of the Algorithm based on Gröbner bases

In this section we propose our new algorithm to compute the cohomology spaces of Lie (super) algebras, based on Proposition 3.3 and Theorem 3.1. This section includes also an example which illustrates the behavior of this algorithm.

Algorithm 1 "LSAC"

Require: $\begin{cases} \mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} & : \text{ an } m\text{-dimensional Lie (super) algebra} \\ V & : \text{ an } n\text{-dimensional } \mathfrak{g}\text{-module} \\ k & : \text{ the cohomology order} \end{cases}$

Ensure: $H^k(\mathfrak{g}, V)$;

- Vars $:= \big\{\phi^l_{(i|j)_{r,k}}, \psi^l_{(i|j)_{r,k-1}}\big\}$;
- $\prec :=$ a lexicographical ordering on $\mathbb{K}[\text{Vars}]$ with $\phi^l_{(i|j)_{r,k}} \prec \psi^{l'}_{(i'|j')_{r,k-1}}$;
- $\mathsf{Syst}_\phi(\mathscr{Z}^k) :=$ the set of equations $(\partial^k\Phi)\big((\mathsf{e}_i, \mathsf{o}_j)_{s,k+1}\big) = 0$;
- $\mathtt{G}_{\mathscr{Z}^k} :=$ the reduced Gröbner basis of $\big\langle \mathsf{Syst}_\phi(\mathscr{Z}^k)\big\rangle$ with respect to $\prec$;
- $\mathsf{Syst}_{\psi,\phi}(\mathscr{B}^k) :=$ the set of equations $\partial^{k-1}\Psi\big((\mathsf{e}_i, \mathsf{o}_j)_{r,k}\big) = \Phi\big((\mathsf{e}_i, \mathsf{o}_j)_{r,k}\big)$;
- $\mathtt{G}_{\mathscr{B}^k} :=$ the reduced Gröbner basis of $\big\langle \mathsf{Syst}_{\psi,\phi}(\mathscr{B}^k)\big\rangle \cap \mathbb{K}\big[\phi^l_{(i|j)_{r,k}}\big]$;
- $\big\{\upsilon_l^{(i|j)_{r,k}}\big\} :=$ the auxiliary variables with the same order $\prec$ as for $\big\{\phi^l_{(i|j)_{r,k}}\big\}$ and satisfying $\upsilon_l^{(i|j)_{r,k}} \prec \phi^{l'}_{(i'|j')_{r',k'}}$;
- $\mathsf{BilinearForm} := \sum \phi^l_{(i|j)_{r,k}}\, \upsilon_l^{(i|j)_{r,k}}$ the auxiliary bilinear form in the two collections of variables $\phi^{\cdot}_{\cdot}$ and $\upsilon^{\cdot}_{\cdot}$;
- $\mathsf{C}_{\mathscr{Z}^k} := \Big\{\mathrm{Coeff}_{\phi^l_{(i|j)_{r,k}}}\Big(\mathrm{NF}_{\mathtt{G}_{\mathscr{Z}^k}}\big(\mathsf{BilinearForm}\big)\Big)\Big\}$;
- $\mathtt{Basis}(\mathscr{Z}^k) :=$ the reduced Gröbner basis of $\mathsf{C}_{\mathscr{Z}^k}$ with respect to $\prec$;
- $\mathsf{C}_{\mathscr{B}^k} := \Big\{\mathrm{Coeff}_{\phi^l_{(i|j)_{r,k}}}\Big(\mathrm{NF}_{\mathtt{G}_{\mathscr{B}^k}}\big(\mathsf{BilinearForm}\big)\Big)\Big\}$;
- $\mathtt{Basis}(\mathscr{B}^k) :=$ the reduced Gröbner basis of $\mathsf{C}_{\mathscr{B}^k}$ with respect to $\prec$;

Return: $\mathtt{Basis}\big(\mathscr{Z}^k/\mathscr{B}^k\big) :=$ the reduced Gröbner basis of:

$$\Big\langle\, \mathrm{NF}_{\mathtt{Basis}(\mathscr{B}^k)}\big(\vartheta\big) : \vartheta \in \mathtt{Basis}(\mathscr{Z}^k)\,\Big\rangle.$$

Example 4.1. Consider the 2-dimensional vector space $V = \mathbb{C}e_0 \oplus \mathbb{C}e_1$ and 4-dimensional Lie super algebra $\mathfrak{g} := \mathfrak{gl}(1|1) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_0 = \mathbb{C}\mathsf{c} \oplus \mathbb{C}\mathsf{h}$ and $\mathfrak{g}_1 = \mathbb{C}\mathsf{x} \oplus \mathbb{C}\mathsf{y}$, with the following table of commutators:

	c	h	x	y
c	0	0	0	0
h	$*$	0	x	$-$y
x	$*$	$*$	0	c
y	$*$	$*$	$*$	0,

and with the following action of $\mathfrak{g}$ on V:

	e_0	e_1
c	0	0
h	$2e_0$	e_1
x	0	0
y	e_1	0.

We aim to compute the second cohomology space $H^2(\mathfrak{g}, V)$. First computations give the following two required systems:

$$\mathtt{G}_{\mathscr{Z}^2} = \Big\{\phi^{e_1}_{x,x},\ \phi^{e_0}_{y,y},\ -\phi^{e_1}_{c,y} + \phi^{e_0}_{x,y},\ -2\phi^{e_1}_{c,x} + \phi^{e_0}_{x,x},\ -3\phi^{e_1}_{y,y} + 2\phi^{e_0}_{h,y},\\ \phi^{e_1}_{c,h} - \phi^{e_1}_{x,y} + \phi^{e_0}_{h,x},\ \phi^{e_0}_{c,y},\ \phi^{e_0}_{c,x},\ -2\phi^{e_1}_{c,y} + \phi^{e_0}_{c,h}\Big\},$$

$$\mathtt{G}_{\mathscr{B}^2} = \Big\{\phi^{e_1}_{x,x},\ \phi^{e_1}_{h,x},\ \phi^{e_1}_{c,x},\ \phi^{e_0}_{y,y},\ -\phi^{e_1}_{c,y} + \phi^{e_0}_{x,y},\ \phi^{e_0}_{x,x},\ -3\phi^{e_1}_{y,y} + 2\phi^{e_0}_{h,y},\\ \phi^{e_1}_{c,h} - \phi^{e_1}_{x,y} + \phi^{e_0}_{h,x},\ \phi^{e_0}_{c,y},\ \phi^{e_0}_{c,x},\ -2\phi^{e_1}_{c,y} + \phi^{e_0}_{c,h}\Big\}.$$

In the next step, we collect the coefficients of the variables $\phi^l_{(i|j)_{r,k}}$ in the normal form of the corresponding bilinear form $\sum \phi^l_{(i|j)_{r,k}} v_l^{(i|j)_{r,k}}$ with respect to the above reduced Gröbner bases and we obtain:

$$\begin{aligned}\mathtt{Basis}\big(\mathscr{Z}^2\big) &= \big\{v^{c,h}_{e_1} - v^{h,x}_{e_0}, v^{c,x}_{e_1} + 2v^{x,x}_{e_0}, 2v^{c,h}_{e_0} + v^{x,y}_{e_0} + v^{c,y}_{e_1}, v^{h,x}_{e_1}, v^{h,y}_{e_1}\big\},\\ \mathtt{Basis}\big(\mathscr{B}^2\big) &= \big\{v^{c,h}_{e_1} - v^{h,x}_{e_0}, 2v^{c,h}_{e_0} + v^{x,y}_{e_0} + v^{c,y}_{e_1}, v^{h,y}_{e_1}\big\},\end{aligned}$$

of cardinalities 5 and 3, respectively. Now the last step of the algorithm provides $5 - 3 = 2$ basis elements:

$$\mathtt{Basis}(\mathscr{Z}^2/\mathscr{B}^2) = \big\{v^{h,x}_{e_1}, v^{c,x}_{e_1} + 2v^{x,x}_{e_0}\big\},$$

which correspond to the following basis for $H^2(\mathfrak{g}, V)$ in the notation (5):

$$H^2(\mathfrak{g}, V) = \big\langle \Lambda^{h,x}_{e_1}, \Lambda^{c,x}_{e_1} + 2\Lambda^{x,x}_{e_0}\big\rangle.$$

Example 4.2. Let $\mathfrak{h}$ be the 7-dimensional standard Lie algebra over $\mathbb{Q}$ whose basis elements $\{\mathsf{l}_1, \mathsf{l}_2, \mathsf{d}, \mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3, \mathsf{r}\}$ enjoy the following commutator table ([19]):

	t_1	t_2	t_3	l_1	l_2	r	d
t_1	0	0	0	$-\mathsf{t}_2$	$-\mathsf{t}_3$	0	$2\mathsf{t}_1$
t_2	$*$	0	0	0	0	t_3	$-3\,\mathsf{t}_2$
t_3	$*$	$*$	0	0	0	$-\mathsf{t}_2$	$-3\,\mathsf{t}_3$
l_1	$*$	$*$	$*$	0	t_1	l_2	$-\mathsf{l}_1$
l_2	$*$	$*$	$*$	$*$	0	$-\mathsf{l}_1$	$-\mathsf{l}_2$
r	$*$	$*$	$*$	$*$	$*$	0	0
d	$*$	$*$	$*$	$*$	$*$	$*$	0

and let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{h}$ which is generated by $\{\mathsf{l}_1, \mathsf{l}_2, \mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3\}$. We would like to compute the fourth cohomology space $H^4(\mathfrak{g}, \mathfrak{h})$. Applying the algorithm, a computer yields the reduced Gröbner basis:

$$\mathtt{G}_{\mathscr{Z}^4} = \Big\{\phi^r_{l_1,t_1,t_2,t_3} - \phi^d_{l_2,t_1,t_2,t_3},\ \phi^d_{l_1,t_1,t_2,t_3} + \phi^r_{l_2,t_1,t_2,t_3},\ 2\phi^d_{l_1,l_2,t_2,t_3} - \phi^{l_1}_{l_1,t_1,t_2,t_3} - \phi^{l_2}_{l_2,t_1,t_2,t_3},\\ \phi^r_{l_1,l_2,t_1,t_2} - 3\phi^d_{l_1,l_2,t_1,t_3} + \phi^{l_1}_{l_1,l_2,t_2,t_3} - \phi^{t_1}_{l_2,t_1,t_2,t_3},\\ 3\phi^d_{l_1,l_2,t_1,t_2} + \phi^r_{l_1,l_2,t_1,t_3} + \phi^{l_2}_{l_1,l_2,t_2,t_3} + \phi^{t_1}_{l_1,t_1,t_2,t_3}\Big\},$$

together with:

$$
\begin{aligned}
\mathtt{G}_{\mathscr{B}^4} = \Big\{ & \phi^r_{l_2,t_1,t_2,t_3},\ \phi^d_{l_2,t_1,t_2,t_3},\ \phi^r_{l_1,t_1,t_2,t_3},\ \phi^d_{l_1,t_1,t_2,t_3},\ \phi^{l_2}_{l_1,t_1,t_2,t_3} + \phi^{l_1}_{l_2,t_1,t_2,t_3}, \\
& - \phi^{l_2}_{l_2,t_1,t_2,t_3} + \phi^{l_1}_{l_1,t_1,t_2,t_3},\ - \phi^{l_1}_{l_2,t_1,t_2,t_3} + \phi^r_{l_1,l_2,t_2,t_3},\ -\phi^{l_2}_{l_2,t_1,t_2,t_3} + \phi^d_{l_1,l_2,t_2,t_3}, \\
& \phi^r_{l_1,l_2,t_1,t_2} - 3\phi^d_{l_1,l_2,t_1,t_3} + \phi^{l_1}_{l_1,l_2,t_2,t_3} - \phi^{t_1}_{l_2,t_1,t_2,t_3}, \\
& 3\phi^d_{l_1,l_2,t_1,t_2} + \phi^r_{l_1,l_2,t_1,t_3} + \phi^{l_2}_{l_1,l_2,t_2,t_3} + \phi^{t_1}_{l_1,t_1,t_2,t_3} \Big\}.
\end{aligned}
$$

Now, collecting the coefficients of the variables $\phi^l_{(i|j)_{r,k}}$ in the normal form of $\sum \phi^l_{(i|j)_{r,k}} v_l^{(i|j)_{r,k}}$ with respect to $\mathtt{G}_{\mathscr{Z}^4}$ and $\mathtt{G}_{\mathscr{B}^4}$, we obtain:

$$
\begin{aligned}
& \mathtt{Basis}(\mathscr{Z}^4) = \big\{ v^{l_2,t_1,t_2,t_3}_{t_3},\ v^{l_2,t_1,t_2,t_3}_{t_2},\ v^{l_2,t_1,t_2,t_3}_{l_1},\ v^{l_1,t_1,t_2,t_3}_{r} + v^{l_2,t_1,t_2,t_3}_{d},\ v^{l_1,t_1,t_2,t_3}_{t_3},\ v^{l_1,t_1,t_2,t_3}_{t_2}, \\
& v^{l_1,t_1,t_2,t_3}_{d} - v^{l_2,t_1,t_2,t_3}_{r},\ v^{l_1,t_1,t_2,t_3}_{l_2},\ -v^{l_2,t_1,t_2,t_3}_{l_2} + v^{l_1,t_1,t_2,t_3}_{l_1},\ v^{l_1,l_2,t_2,t_3}_{r},\ v^{l_1,l_2,t_2,t_3}_{t_3},\ v^{l_1,l_2,t_2,t_3}_{t_2}, \\
& v^{l_1,l_2,t_2,t_3}_{t_1},\ v^{l_1,l_2,t_2,t_3}_{d} + 2v^{l_2,t_1,t_2,t_3}_{l_2},\ -v^{l_1,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_2,t_3}_{l_2},\ v^{l_1,l_2,t_2,t_3}_{l_1} + v^{l_2,t_1,t_2,t_3}_{t_1}, \\
& - v^{l_1,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_1,t_3}_{r},\ v^{l_1,l_2,t_1,t_3}_{t_3},\ v^{l_1,l_2,t_1,t_3}_{t_2},\ v^{l_1,l_2,t_1,t_3}_{t_1},\ -3v^{l_2,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_1,t_3}_{d}, \\
& v^{l_1,l_2,t_1,t_3}_{l_2},\ v^{l_1,l_2,t_1,t_3}_{l_1},\ v^{l_2,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_1,t_2}_{r},\ v^{l_1,l_2,t_1,t_2}_{t_3},\ v^{l_1,l_2,t_1,t_2}_{t_2},\ v^{l_1,l_2,t_1,t_2}_{t_1}, \\
& - 3v^{l_1,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_1,t_2}_{d},\ v^{l_1,l_2,t_1,t_2}_{l_2},\ v^{l_1,l_2,t_1,t_2}_{l_1} \big\},
\end{aligned}
$$

$$
\begin{aligned}
& \mathtt{Basis}(\mathscr{B}^4) = \big\{ v^{l_2,t_1,t_2,t_3}_{t_3},\ v^{l_2,t_1,t_2,t_3}_{t_2},\ v^{l_1,t_1,t_2,t_3}_{t_3},\ v^{l_1,t_1,t_2,t_3}_{t_2},\ -v^{l_1,t_1,t_2,t_3}_{l_2} + v^{l_1,l_2,t_2,t_3}_{r} + v^{l_2,t_1,t_2,t_3}_{l_1}, \\
& v^{l_1,l_2,t_2,t_3}_{t_3},\ v^{l_1,l_2,t_2,t_3}_{t_2},\ v^{l_1,l_2,t_2,t_3}_{t_1},\ v^{l_1,t_1,t_2,t_3}_{l_1} + v^{l_1,l_2,t_2,t_3}_{d} + v^{l_2,t_1,t_2,t_3}_{l_2},\ -v^{l_1,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_2,t_3}_{l_2}, \\
& v^{l_1,l_2,t_2,t_3}_{l_1} + v^{l_2,t_1,t_2,t_3}_{t_1},\ -v^{l_1,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_1,t_3}_{r},\ v^{l_1,l_2,t_1,t_3}_{t_3},\ v^{l_1,l_2,t_1,t_3}_{t_2},\ v^{l_1,l_2,t_1,t_3}_{t_1}, \\
& - 3v^{l_2,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_1,t_3}_{d},\ v^{l_1,l_2,t_1,t_3}_{l_2},\ v^{l_1,l_2,t_1,t_3}_{l_1},\ v^{l_2,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_1,t_2}_{r},\ v^{l_1,l_2,t_1,t_2}_{t_3}, \\
& v^{l_1,l_2,t_1,t_2}_{t_2},\ v^{l_1,l_2,t_1,t_2}_{t_1},\ -3v^{l_1,t_1,t_2,t_3}_{t_1} + v^{l_1,l_2,t_1,t_2}_{d},\ v^{l_1,l_2,t_1,t_2}_{l_2},\ v^{l_1,l_2,t_1,t_2}_{l_1} \big\},
\end{aligned}
$$

of cardinalities 30 and 25, respectively. The last step provides a basis of $5 = 30 - 25$ vectors for $\mathscr{Z}^4/\mathscr{B}^4$ represented by means of the following 5 associated linear forms:

$$
\begin{aligned}
\mathtt{Basis}\big(\mathscr{Z}^4/\mathscr{B}^4\big) = \big\{ & v^{l_2,t_1,t_2,t_3}_{l_1},\ v^{l_1,t_1,t_2,t_3}_{r} + v^{l_2,t_1,t_2,t_3}_{d},\ v^{l_1,t_1,t_2,t_3}_{d} - v^{l_2,t_1,t_2,t_3}_{r}, \\
& v^{l_1,t_1,t_2,t_3}_{l_2},\ -v^{l_2,t_1,t_2,t_3}_{l_2} + v^{l_1,t_1,t_2,t_3}_{l_1} \big\}.
\end{aligned}
$$

Coming back to the notation (5), this means that the desired fourth cohomology space $H^4(\mathfrak{g}, \mathfrak{h})$ is 5-dimensional with the following generators:

$$
\begin{aligned}
H^4\big(\mathfrak{g}, \mathfrak{h}\big) = \big\{ & \Lambda^{l_2,t_1,t_2,t_3}_{l_1},\ \Lambda^{l_1,t_1,t_2,t_3}_{r} + \Lambda^{l_2,t_1,t_2,t_3}_{d},\ \Lambda^{l_1,t_1,t_2,t_3}_{l_2},\ \Lambda^{l_1,t_1,t_2,t_3}_{d} - \Lambda^{l_2,t_1,t_2,t_3}_{r}, \\
& - \Lambda^{l_2,t_1,t_2,t_3}_{l_2} + \Lambda^{l_1,t_1,t_2,t_3}_{l_1} \big\},
\end{aligned}
$$

More explicitly, one can rewrite these generators as follows:

$$
\begin{aligned}
H^4\big(\mathfrak{g}, \mathfrak{h}\big) = \big\{ & \mathsf{l}_2^* \wedge \mathsf{t}_1^* \wedge \mathsf{t}_2^* \wedge \mathsf{t}_3^* \otimes \mathsf{l}_1,\ \ \mathsf{l}_1^* \wedge \mathsf{t}_1^* \wedge \mathsf{t}_2^* \wedge \mathsf{t}_3^* \otimes \mathsf{r} + \mathsf{l}_2^* \wedge \mathsf{t}_1^* \wedge \mathsf{t}_2^* \wedge \mathsf{t}_3^* \otimes \mathsf{d}, \\
& \mathsf{l}_1^* \wedge \mathsf{t}_1^* \wedge \mathsf{t}_2^* \wedge \mathsf{t}_3^* \otimes \mathsf{l}_2,\ \ \mathsf{l}_1^* \wedge \mathsf{t}_1^* \wedge \mathsf{t}_2^* \wedge \mathsf{t}_3^* \otimes \mathsf{d} - \mathsf{l}_2^* \wedge \mathsf{t}_1^* \wedge \mathsf{t}_2^* \wedge \mathsf{t}_3^* \otimes \mathsf{r}, \\
& - \mathsf{l}_2^* \wedge \mathsf{t}_1^* \wedge \mathsf{t}_2^* \wedge \mathsf{t}_3^* \otimes \mathsf{l}_2 + \mathsf{l}_1^* \wedge \mathsf{t}_1^* \wedge \mathsf{t}_2^* \wedge \mathsf{t}_3^* \otimes \mathsf{l}_1 \big\}.
\end{aligned}
$$

5. Improvement of the Algorithm when Cohomology Spaces Split

As we saw, the two collections of Cartesian linear equations $\mathsf{Syst}_\phi(\mathscr{Z}^k)$ and $\mathsf{Syst}_\phi(\mathscr{Z}^k)$ have an essential rôle in the process, and if the number of variables in them increases, one can expect that the complexity of computations will increases too. Here, in the case of standard Lie algebras $\mathfrak{g} \subset \mathfrak{h} = V$, one further aim could to set up a refined algorithm which inspects whether these equations split up into a collection of sub-equations each of which involves a smaller number of variables. However, this kind of problem lies a bit outside the scope of the present article, closer to plain searching-and-listing algorithmic procedures, because it amounts to read, by means of a computer, some two given systems of linear equations in some variables $(x_1, \dots, x_n)$ and to pick up step by step the appearing nonzero $\lambda_i \, x_i$ until one gathers pairs of collections of equations which involve only a *subset* of variables, all subsets being pairwise distinct.

Nevertheless, the circumstance of spitting up naturally occurs for instance when the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are *graded* at the beginning, in the sense of Tanaka ([21, 1, 2]), namely when one has decompositions into direct sums of $\mathbb{K}$-vector subspaces:

$$\begin{aligned}\mathfrak{h} &= \mathfrak{h}_{-a} \oplus \cdots \oplus \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_b \\ \mathfrak{g} &= \mathfrak{h}_{-a} \oplus \cdots \oplus \mathfrak{h}_{-1},\end{aligned}$$

where $a \geqslant 1$ and $b \geqslant 0$ are certain two integers, with the property that:

$$\big[\mathfrak{h}_{\ell_1}, \, \mathfrak{h}_{\ell_2}\big] \subset \mathfrak{h}_{\ell_1+\ell_2},$$

for all $\ell_1, \ell_2 \in \mathbb{Z}$, after prolonging trivially $\mathfrak{h}_\ell := \{0\}$ for either $\ell \leqslant -a-1$ or $\ell \geqslant b+1$. Then each space of k-cochains $\mathscr{C}^k(\mathfrak{g}, \mathfrak{h})$ naturally splits up as a direct sum of so-called *homogeneous k-cochains* as follows: a k-cochain $\Phi \in \mathscr{C}^k(\mathfrak{g}, \mathfrak{h})$ is said to be *of homogeneity* a certain integer $h \in \mathbb{Z}$ whenever for any k vectors:

$$\mathsf{z}_{i_1} \in \mathfrak{h}_{\ell_1}, \dots\dots, \mathsf{z}_{i_k} \in \mathfrak{h}_{\ell_k}$$

belonging to certain arbitrary but determined $\mathfrak{h}$-components, its value:

$$\Phi(\mathsf{z}_{i_1}, \dots, \mathsf{z}_{i_k}) \in \mathfrak{h}_{\ell_1+\cdots+\ell_k+h}$$

belongs to the $(\ell_1 + \cdots + \ell_k + h)$-th component of $\mathfrak{h}$. Then one easily convinces oneself (*see* also [11]) that any k-cochain $\Phi \in \mathscr{C}^k(\mathfrak{g}, \mathfrak{h})$ splits up as a direct sum of k-cochains of fixed homogeneity:

$$\Phi = \cdots + \Phi^{[h-1]} + \Phi^{[h]} + \Phi^{[h+1]} + \cdots ,$$

where we denote the completely h-homogeneous component of Φ just by $\Phi^{[h]}$. In other words:

$$\mathscr{C}^k(\mathfrak{g}, \mathfrak{h}) = \bigoplus_{h \in \mathbb{Z}} \mathscr{C}^k_{[h]}(\mathfrak{g}, \mathfrak{h}),$$

where of course the spaces $\mathscr{C}^k_{[h]}(\mathfrak{g}, \mathfrak{h})$ reduce to $\{0\}$ for all large $|h|$. Furthermore, applying the definition (2), one verifies the important fact that ∂^k respects homogeneity for all $k = 0, 1, \dots, n$, that is to say, for any $h \in \mathbb{Z}$, one has:

$$\partial^k\big(\mathscr{C}^k_{[h]}\big) \subset \mathscr{C}^{k+1}_{[h]},$$

whence the complex (3) splits up as a direct sum of complexes:

$$0 \xrightarrow{\partial^0_{[h]}} \mathscr{C}^1 \xrightarrow{\partial^1_{[h]}} \mathscr{C}^2 \xrightarrow{\partial^2_{[h]}} \cdots \xrightarrow{\partial^{m-2}_{[h]}} \mathscr{C}^{m-1} \xrightarrow{\partial^{m-1}_{[h]}} \mathscr{C}^m \xrightarrow{\partial^m_{[h]}} 0$$

indexed by $h \in \mathbb{Z}$, where $\partial^k_{[h]}$ naturally denotes the restriction:

$$\partial^k_{[h]} := \partial^k\big|_{\mathscr{C}^k_{[h]}} : \mathscr{C}^k_{[h]} \longrightarrow \mathscr{C}^{k+1}_{[h]}.$$

Consequently, one may introduce the spaces of h-*homogeneous cocycles* of order k:

$$\mathscr{Z}^k_{[h]}(\mathfrak{g},\mathfrak{h}) := \ker\big(\partial^k_{[h]} \colon \mathscr{C}^k_{[h]} \to \mathscr{C}^{k+1}_{[h]}\big),$$

together with the spaces of h-*homogeneous coboundaries* of order k:

$$\mathscr{B}^k_{[h]}(\mathfrak{g},\mathfrak{h}) := \mathrm{im}\big(\partial^{k-1}_{[h]} \colon \mathscr{C}^{k-1}_{[h]} \to \mathscr{C}^{k}_{[h]}\big).$$

The computation of the h-homogeneous k-th cohomology spaces:

$$H^k_{[h]}(\mathfrak{g},\mathfrak{h}) := \frac{\mathscr{Z}^k_{[h]}(\mathfrak{g},\mathfrak{h})}{\mathscr{B}^k_{[h]}(\mathfrak{g},\mathfrak{h})}$$

then requires to deal with vector (sub)spaces of smaller dimensions and enables one to reconstitute the complete cohomology space:

$$H^k(\mathfrak{g},\mathfrak{g}) = \bigoplus_{h\in\mathbb{Z}} H^k_{[h]}(\mathfrak{g},\mathfrak{g}).$$

Example 5.1. Let $\mathfrak{h}$ be the 8-dimensional Lie algebra over $\mathbb{Q}$ whose basis elements $\{\mathsf{t},\mathsf{h}_1,\mathsf{h}_2,\mathsf{r},\mathsf{d},\mathsf{i}_1,\mathsf{i}_2,\mathsf{j}\}$ enjoy the following commutator table:

	t	h_1	h_2	d	r	i_1	i_2	j
t	0	0	0	$2\,\mathsf{t}$	0	h_1	h_2	d
h_1	$*$	0	$4\,\mathsf{t}$	h_1	h_2	$6\,\mathsf{r}$	$2\,\mathsf{d}$	i_1
h_2	$*$	$*$	0	h_2	$-\mathsf{h}_1$	$-2\,\mathsf{d}$	$6\,\mathsf{r}$	i_2
d	$*$	$*$	$*$	0	0	i_1	i_2	$2\,\mathsf{j}$
r	$*$	$*$	$*$	$*$	0	$-\mathsf{i}_2$	i_1	0
i_1	$*$	$*$	$*$	$*$	$*$	0	$4\,\mathsf{j}$	0
i_2	$*$	$*$	$*$	$*$	$*$	$*$	0	0
j	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0

and let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{h}$ which is generated by $\mathsf{t},\mathsf{h}_1,\mathsf{h}_2$, *see* [1] for application to the differential study of Cartan connection in local Cauchy-Riemann geometry. We want to compute $H^2(\mathfrak{g},\mathfrak{h})$. The geometry provides a natural graduation:

$$\mathfrak{h} = \underbrace{\mathfrak{h}_{-2}\oplus\mathfrak{h}_{-1}}_{\mathfrak{g}} \oplus\mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\mathfrak{h}_2$$

where:

$$\mathfrak{h}_{-2} = \mathbb{R}\,\mathsf{t}, \quad \mathfrak{h}_{-1} = \mathbb{R}\,\mathsf{h}_1\oplus\mathbb{R}\,\mathsf{h}_2, \quad \mathfrak{h}_0 = \mathbb{R}\,\mathsf{d}\oplus\mathbb{R}\,\mathsf{r}, \quad \mathfrak{h}_1 = \mathbb{R}\,\mathsf{i}_1\oplus\mathbb{R}\,\mathsf{i}_2, \quad \mathfrak{h}_2 = \mathbb{R}\,\mathsf{j},$$

and one verifies that the commutator table written above respects this graduation. A general 2-cochain $\Phi \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{h}$ writes under the form:

$$\begin{array}{ll}
& \Phi = \phi_t^{h_1h_2}\, \mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{t} + \quad \boxed{0} \\
\boxed{1} & + \phi_t^{th_1}\, \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{t} + \phi_t^{th_2}\, \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{t} + \phi_{h_1}^{h_1h_2}\, \mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{t} + \phi_{h_2}^{h_1h_2}\, \mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{h}_2 + \\
\boxed{2} & + \phi_{h_1}^{th_1}\, \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{h}_1 + \phi_{h_2}^{th_1}\, \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{h}_2 + \phi_{h_1}^{th_2}\, \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{h}_1 + \phi_{h_2}^{th_2}\, \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{h}_2 + \\
& \qquad + \phi_d^{h_1h_2}\, \mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{d} + \phi_r^{h_1h_2}\, \mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{r} + \\
\boxed{3} & + \phi_d^{th_1}\, \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{d} + \phi_r^{th_1}\, \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{r} + \phi_d^{th_2}\, \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{d} + \phi_r^{th_2}\, \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{r} \\
& \qquad + \phi_{i_1}^{h_1h_2}\, \mathsf{h}_1^* \wedge \mathsf{h}_2{}^* \otimes \mathsf{i}_1 + \phi_{i_2}^{h_1h_2}\, \mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_2 + \\
\boxed{4} & + \phi_{i_1}^{th_1}\, \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{i}_1 + \phi_{i_2}^{th_1}\, \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{i}_2 + \phi_{i_1}^{th_2}\, \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_1 + \phi_{i_2}^{th_2}\, \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_2 \\
& \qquad + \phi_j^{h_1h_2}\, \mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{j} + \\
\boxed{5} & + \phi_j^{th_1}\, \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{j} + \phi_j^{th_2}\, \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{j},
\end{array}$$

where framed numbers denote homogeneity of their lines. After computations, a 2-cochain Φ is a 2-cocycle if and only if its 24 coefficients satisfy the following seven linear equations, ordered line by line by increasing homogeneity:

$$\begin{array}{lll}
\boxed{2} & 0 = 2\phi_d^{h_1h_2} - 4\phi_{h_2}^{th_2} - 4\phi_{h_1}^{th_1}, & \\
\boxed{3} & 0 = \phi_{i_1}^{h_1h_2} - \phi_d^{th_2} - \phi_r^{th_1}, & 0 = \phi_{i_2}^{h_1h_2} - \phi_r^{th_2} + \phi_d^{th_1}, \\
\boxed{4} & 0 = \phi_j^{h_1h_2} - 2\phi_{i_2}^{th_2} - 2\phi_{i_1}^{th_1}, & 0 = -6\phi_{i_1}^{th_2} + 6\phi_{i_2}^{th_1}, \\
\boxed{5} & 0 = -\phi_j^{th_2}, \qquad 0 = \phi_j^{th_1}. &
\end{array}$$

Next, a general 1-cochain $\Psi \in \Lambda^1\mathfrak{g}^* \otimes \mathfrak{h}$ writes under the form:

$$\begin{array}{ll}
& \Psi = \psi_t^{h_1}\, \mathsf{h}_1^* \otimes \mathsf{t} + \psi_t^{h_2}\, \mathsf{h}_2^* \otimes \mathsf{t} + \quad \boxed{-1} \\
\boxed{0} & + \psi_t^t\, \mathsf{t}^* \otimes \mathsf{t} + \psi_{h_1}^{h_1}\, \mathsf{h}_1^* \otimes \mathsf{h}_1 + \psi_{h_2}^{h_1}\, \mathsf{h}_1^* \otimes \mathsf{h}_2 + \psi_{h_1}^{h_2}\, \mathsf{h}_2^* \otimes \mathsf{h}_1 + \psi_{h_2}^{h_2}\, \mathsf{h}_2^* \otimes \mathsf{h}_2 + \\
\boxed{1} & + \psi_{h_1}^t\, \mathsf{t}^* \otimes \mathsf{h}_1 + \psi_{h_2}^t\, \mathsf{t}^* \otimes \mathsf{h}_2 + \psi_d^{h_1}\, \mathsf{h}_1^* \otimes \mathsf{d} + \psi_r^{h_1}\, \mathsf{h}_1^* \otimes \mathsf{r} + \psi_d^{h_2}\, \mathsf{h}_2^* \otimes \mathsf{d} + \psi_r^{h_2}\, \mathsf{h}_2^* \otimes \mathsf{r} + \\
\boxed{2} & + \psi_d^t\, \mathsf{t}^* \otimes \mathsf{d} + \psi_r^t\, \mathsf{t}^* \otimes \mathsf{r} + \psi_{i_1}^{h_1}\, \mathsf{h}_1^* \otimes \mathsf{i}_1 + \psi_{i_2}^{h_1}\, \mathsf{h}_1^* \otimes \mathsf{i}_2 + \psi_{i_1}^{h_2}\, \mathsf{h}_2^* \otimes \mathsf{i}_1 + \psi_{i_2}^{h_2}\, \mathsf{h}_2^* \otimes \mathsf{i}_2 + \\
\boxed{3} & + \psi_{i_1}^t\, \mathsf{t}^* \otimes \mathsf{i}_1 + \psi_{i_2}^t\, \mathsf{t}^* \otimes \mathsf{i}_2 + \psi_j^{h_1}\, \mathsf{h}_1^* \otimes \mathsf{j} + \psi_j^{h_2}\, \mathsf{h}_2^* \otimes \mathsf{j} + \\
\boxed{4} & + \psi_j^t\, \mathsf{t}^* \otimes \mathsf{j}.
\end{array}$$

The condition that $\Phi = \partial^1\Psi$ then reads in homogeneous-decomposed form:

$$\begin{array}{llllll}
\boxed{1} & \phi_t^{th_1} = 2\psi_d^{h_1} - 4\psi_{h_2}^t & \boxed{1} & \phi_t^{th_2} = 2\psi_d^{h_2} + 4\psi_{h_1}^t & \boxed{0} & \phi_t^{h_1h_2} = 4\psi_{h_2}^{h_2} + 4\psi_{h_1}^{h_1} - 4\psi_t^t \\
\boxed{2} & \phi_{h_1}^{th_1} = \psi_{i_1}^{h_1} - \psi_d^t & \boxed{2} & \phi_{h_1}^{th_2} = \psi_{i_1}^{h_2} + \psi_r^t & \boxed{1} & \phi_{h_1}^{h_1h_2} = \psi_d^{h_2} + \psi_r^{h_1} - 4\psi_{h_1}^t \\
\boxed{2} & \phi_{h_2}^{th_1} = \psi_{i_2}^{h_1} - \psi_r^t & \boxed{2} & \phi_{h_2}^{th_2} = \psi_{i_2}^{h_2} - \psi_d^t & \boxed{1} & \phi_{h_2}^{h_1h_2} = \psi_r^{h_2} - \psi_d^{h_1} + 4\psi_{h_2}^t \\
\boxed{3} & \phi_d^{th_1} = \psi_j^{h_1} - 2\psi_{i_2}^t & \boxed{3} & \phi_d^{th_2} = \psi_j^{h_2} + 2\psi_{i_1}^t & \boxed{2} & \phi_d^{h_1h_2} = 2\psi_{i_2}^{h_2} + 2\psi_{i_1}^{h_1} - 4\psi_d^t \\
\boxed{3} & \phi_r^{th_1} = -6\psi_{i_1}^t & \boxed{3} & \phi_r^{th_2} = -6\psi_{i_2}^t & \boxed{2} & \phi_r^{h_1h_2} = 6\psi_{i_1}^{h_2} - 6\psi_{i_2}^{h_1} - 4\psi_r^t \\
\boxed{4} & \phi_{i_1}^{th_1} = -\psi_j^t & \boxed{4} & \phi_{i_1}^{th_2} = 0 & \boxed{3} & \phi_{i_1}^{h_1h_2} = \psi_j^{h_2} - 4\psi_{i_1}^t \\
\boxed{4} & \phi_{i_2}^{th_1} = 0 & \boxed{4} & \phi_{i_2}^{th_2} = -\psi_j^t & \boxed{3} & \phi_{i_2}^{h_1h_2} = -\psi_j^{h_1} - 4\psi_{i_2}^t \\
\boxed{5} & \phi_j^{th_1} = 0 & \boxed{5} & \phi_j^{th_2} = 0 & \boxed{4} & \phi_j^{h_1h_2} = -4\psi_j^t.
\end{array}$$

One can then apply our algorithm to each subcollection of equations for every fixed homogeneity, and find that $H^2(\mathfrak{g}, \mathfrak{h})$ is 2-dimensional, generated by:

$$\boxed{\begin{array}{ll} & \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_2 - 2\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{j} \\ \text{and:} & \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_1 - \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{i}_2, \end{array}}$$

with the further observation that all cohomologies are zero except in homogeneity 4:

Homogeneity	$\dim \mathscr{C}^2$	$\dim \mathscr{Z}^2$	$\dim \mathscr{B}^2$	$\dim H^2$
0	1	1	1	0
1	4	4	4	0
2	6	5	5	0
3	6	4	4	0
4	5	3	1	2
5	2	0	0	0

To conclude the presentation, in the next table, we present the speediness of the algorithm for our two Examples 4.2 and 5.1, and also for $H^k(\mathfrak{gl}(3), \mathfrak{sl}(3))$:

Cohomology	Order	time(sec.)	memory(M)	$\dim(\mathscr{C}^k)$	$\dim(\mathscr{Z}^k)$	$\dim(\mathscr{B}^k)$	$\dim(H^k)$
Example 4.1	2	0.0	0.23	16	6	4	2
Example 4.2	2	0.125	3.6	70	25	33	8
Example 4.2	3	0.125	4.3	70	37	45	8
Example 4.2	4	0.03	1.4	35	25	30	5
Example 4.2	5	0.0	0.16	7	5	7	2
Example 5.1	2	0.015	0.7	24	15	17	2
Example 5.1	3	0.0	0.18	8	7	8	1
$(\mathfrak{gl}(3), \mathfrak{sl}(3))$	2	2	8.6	252	64	64	0
$(\mathfrak{gl}(3), \mathfrak{sl}(3))$	3	24	40	504	188	189	1

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DEPARTMENT OF MATHEMATICAL SCIENCES, ISFAHAN UNIVERSITY OF TECHNOLOGY, ISFAHAN, 84156-8311, IRAN
E-mail address: m.aghasi@cc.iut.ac.ir

DEPARTMENT OF MATHEMATICAL SCIENCES, ISFAHAN UNIVERSITY OF TECHNOLOGY, ISFAHAN, 84156-8311, IRAN
E-mail address: b.alizadeh@math.iut.ac.ir

DÉPARTEMENT DE MATHÉMATIQUES D'ORSAY, BÂTIMENT 425, FACULTÉ DES SCIENCES, UNIVERSITÉ PARIS XI - ORSAY, F-91405 ORSAY CEDEX, FRANCE
E-mail address: merker@dma.ens.fr

DEPARTMENT OF MATHEMATICAL SCIENCES, ISFAHAN UNIVERSITY OF TECHNOLOGY, ISFAHAN, 84156-8311, IRAN
E-mail address: sabzevari@math.iut.ac.ir