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Joël Merker, Pawel Nurowski. New Explicit Lorentzian Einstein-Weyl Structures in 3-Dimensions. *Symmetry, Integrability and Geometry: Methods and Applications*, 2020, 16, pp.056. 10.3842/sigma.2020.056 . hal-03286283

HAL Id: hal-03286283

<https://universite-paris-saclay.hal.science/hal-03286283>

Submitted on 14 Jul 2021

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New Explicit Lorentzian Einstein–Weyl Structures in 3-Dimensions

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Received March 30, 2020, in final form June 08, 2020; Published online June 17, 2020

<https://doi.org/10.3842/SIGMA.2020.056>

Abstract. On a 3D manifold, a *Weyl geometry* consists of pairs $(g, A) = (\text{metric}, 1\text{-form})$ modulo gauge $\hat{g} = e^{2\varphi}g$, $\hat{A} = A + d\varphi$. In 1943, Cartan showed that every solution to the Einstein–Weyl equations $R_{(\mu\nu)} - \frac{1}{3}Rg_{\mu\nu} = 0$ comes from an appropriate 3D leaf space quotient of a 7D connection bundle associated with a 3rd order ODE $y''' = H(x, y, y', y'')$ modulo point transformations, provided 2 among 3 primary point invariants vanish

$$\text{Wünschmann}(H) \equiv 0 \equiv \text{Cartan}(H).$$

We find that point equivalence of a single PDE $z_y = F(x, y, z, z_x)$ with para-CR integrability $DF := F_x + z_x F_z \equiv 0$ leads to a *completely similar* 7D Cartan bundle and connection. Then magically, the (complicated) equation $\text{Wünschmann}(H) \equiv 0$ becomes

$$0 \equiv \text{Monge}(F) := 9F_{pp}^2 F_{ppppp} - 45F_{pp} F_{ppp} F_{pppp} + 40F_{ppp}^3, \quad p := z_x,$$

whose solutions are just conics in the $\{p, F\}$ -plane. As an ansatz, we take

$$F(x, y, z, p) := \frac{\alpha(y)(z - xp)^2 + \beta(y)(z - xp)p + \gamma(y)(z - xp) + \delta(y)p^2 + \varepsilon(y)p + \zeta(y)}{\lambda(y)(z - xp) + \mu(y)p + \nu(y)},$$

with 9 arbitrary functions α, \dots, ν of y . This F satisfies $DF \equiv 0 \equiv \text{Monge}(F)$, and we show that the condition $\text{Cartan}(H) \equiv 0$ passes to a certain $K(F) \equiv 0$ which holds for any choice of $\alpha(y), \dots, \nu(y)$. Descending to the leaf space quotient, we gain ∞ -dimensional *functionally parametrized and explicit* families of Einstein–Weyl structures $[(g, A)]$ in 3D. These structures are nontrivial in the sense that $dA \not\equiv 0$ and $\text{Cotton}([g]) \not\equiv 0$.

Key words: Einstein–Weyl structures; Lorentzian metrics; para-CR structures; third-order ordinary differential equations; Monge invariant; Wünschmann invariant; Cartan's method of equivalence; exterior differential systems

2020 Mathematics Subject Classification: 83C15; 53C25; 83C20; 53C25; 53C10; 53C25; 53A30; 53A55; 34A26; 34C14; 58A15; 53-08

1 Introduction

On an n -manifold M , a *Weyl geometry* is a pair (g, A) of a signature $(k, n-k)$ pseudo-Riemannian metric modulo $\hat{g} = e^{2\varphi}g$ together with a 1-form A modulo $\hat{A} = A + d\varphi$, where $\varphi: M \rightarrow \mathbb{R}$ is any function. As in Riemannian geometry, a symmetric Ricci tensor $R_{(\mu\nu)}$ with scalar curvature R

can be defined (see [3, 7, 8] or Section 2). The *Einstein–Weyl equations* in vacuum

$$R_{(\mu\nu)} - \frac{1}{n}Rg_{\mu\nu} = 0, \quad 1 \leq \mu, \nu \leq n, \quad (1.1)$$

which depend only on the class $[(g, A)]$, have raised interest, specially in dimension $n = 3$. We find various functionally parametrized explicit families of solutions. On $\mathbb{R}^3 \ni (x, y, z)$, take for instance 5 free arbitrary functions b, c, k, l, m of y with derivatives b', k' .

Theorem 1.1. *All pairs (g, A) such that*

$$\begin{aligned} g &:= (k + bz)^2 dx^2 + x^2(l^2 - cm) dy^2 + x^2 b^2 dz^2 \\ &\quad + 2x(ckz - blz + kl - bm) dx dy - 2xb(k + bz) dx dz - 2x^2(ck - bl) dy dz, \\ A &:= \frac{-ck + bl + b'k - bk'}{x(ck^2 - 2bkl + b^2m)} (x b dz - (k + bz) dx) \\ &\quad + \frac{bl^2 - cbm - b'kl + bb'm + ckk' - bk'l}{ck^2 - 2bkl + b^2m} dy, \end{aligned}$$

satisfy equations (1.1), hence define a Lorentzian Einstein–Weyl structure on \mathbb{R}^3 .

Moreover, all such examples are generically conformally non-flat, and each of the 5 independent components of the Cotton tensor of the underlying conformal structure $(M, [g])$ is not identically zero.

We discover in fact even more general explicit families of solutions depending on 9 free arbitrary functions of 1 variable y . Explicit examples of Einstein–Weyl structures in 3D were known before [1, 3, 4, 5, 7, 8, 10, 11, 12, 17, 18, 19, 20].

According to [3], all Einstein–Weyl structures may be constructed by a certain quotient process from a 7D Cartan bundle associated with equivalences of 3rd order ordinary differential equations. Those, in turn, are known to be para-CR structures of type $(1, 1, 2)$, cf. [9, Section 5.1.3].

In the present paper, we explore the observation that PDEs on the plane (x, y) of the form $z_y = F(x, y, z, z_x)$, considered modulo point transformations, also happen to be $(1, 1, 2)$ para-CR structures, in certain circumstances. In Section 8, we show how equivalence classes of $(1, 1, 2)$ para-CR structures associated to PDEs $z_y = F$ are ‘embedded’ into the space of equivalence classes of 3rd order ODEs. This distinguishes a certain class of 3rd order ODEs from which we construct our explicit solutions to the Einstein–Weyl equations.

Thus, our main approach is to study point equivalences of a single PDE of the form (novelty)

$$z_y = F(x, y, z, z_x),$$

with unknown $z = z(x, y)$. From para-CR geometry [9, 13], an integrability condition is required, namely,

$$DF := F_x + z_x F_z \equiv 0.$$

To exclude trivial PDEs, another point invariant condition must be assumed:

$$F_{pp} \neq 0 \quad (\text{abbreviate } p := z_x).$$

In Theorem 5.2, we construct a 7-dimensional Cartan bundle/connection $P_7 \rightarrow J_4 \ni (x, y, z, p)$ canonically associated to point equivalences of such PDEs $z_y = F(x, y, z, z_x)$, we determine a canonical coframe $\{\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3\}$ on P_7 , and we find that its structure equations (4.4) incorporate exactly 3 primary invariants, named A_1, B_1, C_1 .

Quite unexpectedly, we realize that these structure equations have the same form as the structure equations of the canonical 7-dimensional Cartan bundle/connection associated with point equivalences of 3rd order ODEs $y''' = H(x, y, y', y'')$. Furthermore, it is known that quite similarly, 3 primary differential invariants govern such geometries. Two among them are: the *Wünschmann invariant* $\mathbf{W}(H)$ [22] and the *Cartan invariant* $\mathbf{C}(H)$ [2, 3]. Since Cartan 1943, it is also known [3, 6, 7, 8, 10] that *all* solutions to the Einstein–Weyl structure equations (1.1) can be obtained from ODEs satisfying $\mathbf{W}(H) \equiv 0 \equiv \mathbf{C}(H)$. Translating what is known for ODEs or performing computations from scratch, we will set up and state Cartan’s construction *from the PDE side*, see Theorem 5.3.

But from the ODE side unfortunately, it is quite difficult to solve Wünschmann’s nonlinear equation incorporating 25 differential monomials

$$\begin{aligned} 0 \equiv \mathbf{W}(H) := & -18qH_qH_{pq} + 9pH_yH_{qq} + 18qHH_{pqq} + 9qH_pH_{qq} - 18pH_qH_{yq} + 18pHH_{yqq} \\ & - 9HH_qH_{qq} + 18pqH_{ypq} + 18pH_{xyq} + 18qH_{xpq} + 9H_xH_{qq} + 18HH_{xqq} \\ & - 18H_qH_{xq} + 18H_pH_q + 9H_{xxq} - 27H_{xp} + 4H_q^3 + 9p^2H_{yyq} - 27pH_{yp} \\ & + 9qH_{yq} + 9q^2H_{ppq} - 27qH_{pp} - 18HH_{pq} + 9H^2H_{qqq} + 54H_y. \end{aligned}$$

This inspired us to try to work on the PDE side $z_y = F(x, y, z, z_x)$, instead of the ODE side. Then *magically*, $\mathbf{W}(H) \equiv 0$ transforms into the much simpler classical invariant of Monge [16]

$$0 \equiv \text{Monge}(F) := 9F_{pp}^2F_{pppp} - 45F_{pp}F_{ppp}F_{pppp} + 40F_{ppp}^3,$$

When $F_{pp} \neq 0$, it is known that $\mathbf{M}(F) \equiv 0$ holds if and only if there exist functions A, B, C, K, L, M of (x, y, z) such that

$$0 \equiv AF^2 + 2BFp + Cp^2 + 2KF + 2Lp + M.$$

Assuming $A := 0$, we obtain the following

Proposition 1.2. *The general solution $F = F(x, y, z, p)$ to*

$$\begin{aligned} 0 &\equiv F_x + pF_z, \\ 0 &\equiv 0 + 2BFp + Cp^2 + 2KF + 2Lp + M \end{aligned}$$

is

$$F = \frac{\alpha(y)(z - xp)^2 + \beta(y)(z - xp)p + \gamma(y)(z - xp) + \delta(y)p^2 + \varepsilon(y)p + \zeta(y)}{\lambda(y)(z - xp) + \mu(y)p + \nu(y)},$$

with 9 arbitrary functions $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \lambda, \mu, \nu$ of y .

Of course, to the Cartan invariant $\mathbf{C}(H)$ from the ODE side there corresponds from the PDE side a certain invariant we name $\mathbf{K}(F)$: its expression appears in Theorem 5.2. Miraculously, then, a direct calculation shows that no further constraint is imposed.

Proposition 1.3. *For any choice of $\alpha(y), \beta(y), \gamma(y), \delta(y), \varepsilon(y), \zeta(y), \lambda(y), \mu(y), \nu(y)$, the second condition*

$$\mathbf{K}(F_{\alpha, \dots, \nu}) \equiv 0$$

for obtaining Weyl pairs $[(g, A)]$ satisfying the Einstein–Weyl field equations (1.1) holds automatically.

We then get – quite long – formulas for pairs $[(g, A)]$ expressed explicitly in terms of $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \lambda, \mu, \nu$. The subfamily for which $\beta = 0, \delta = 0, \varepsilon = 0, \mu = 0$ corresponds (with different notations) to Theorem 1.1.

Theorem 1.4. *Same conclusion as in Theorem 1.1 with*

$$g := \tau^1 \tau^2 + \tau^2 \tau^1 + \tau^3 \tau^3,$$

$$A := \tau^3 \frac{1}{2\Pi} (\gamma\lambda x - \gamma\mu + x\lambda\nu' + \beta\lambda z + \lambda\mu'z - 2\alpha\mu z \\ - \lambda'\mu z - \mu\nu' - x\lambda'\nu - 2x\alpha\nu + \beta\nu + \mu'\nu),$$

with the coframe

$$\tau^1 := dx + \frac{dy}{x\lambda - \mu} (x\beta - \delta - x^2\alpha),$$

$$\tau^2 := \frac{2dy}{x\lambda - \mu} \Pi,$$

$$\tau^3 := (-\lambda z - \nu) dx + \frac{1}{x\lambda - \mu} dy (-\varepsilon\mu + 2x^2\alpha\nu + x\gamma\mu - 2x\beta\nu - \beta\mu z + 2\delta\lambda z + 2x\alpha\mu z \\ + x\varepsilon\lambda + 2\delta\nu - x^2\gamma\lambda - x\beta\lambda z) + (x\lambda - \mu) dz,$$

and the function

$$\Pi := x^2\zeta\lambda^2 + \alpha\mu^2z^2 + 2x\alpha\mu\nu z + x^2\alpha\nu^2 - \beta\lambda\mu z^2 - x\beta\lambda\nu z + \delta\lambda^2z^2 + x\varepsilon\lambda^2z - 2x\zeta\lambda\mu \\ - \beta\mu\nu z - x\beta\nu^2 + 2\delta\lambda\nu z - \varepsilon\lambda\mu z + x\varepsilon\lambda\nu - x\gamma\lambda\mu z - x^2\gamma\lambda\nu + \zeta\mu^2 + \delta\nu^2 - \varepsilon\mu\nu \\ + \gamma\mu^2z + x\gamma\mu\nu,$$

again with $dA \neq 0$ and $\text{Cotton}([g]) \neq 0$.

At the end, we also present other families of functionally parametrized solutions, when $A \neq 0$.

2 Weyl geometry: a summary

In Einstein's theory, gravity is described in terms of a (pseudo-)riemannian metric g called the *gravitational potential*. In Maxwell's theory, the electromagnetic field is described in terms of a 1-form A called the *Maxwell potential*.

In his attempt *Raum, Zeit, Materie* [21] of unifying gravitation and electromagnetism, Weyl was inspired to introduce the synthetic geometric structure on any n -dimensional manifold M^n which consists of classes of such pairs $[(g, A)]$ under the equivalence relation

$$(g, A) \sim (\widehat{g}, \widehat{A})$$

holding by definition if and only if there exists a function $\varphi: M \rightarrow \mathbb{R}$ such that

- (1) $\widehat{g} = e^{2\varphi}g$;
- (2) $\widehat{A} = A + d\varphi$.

Clearly, the electromagnetic field strength $F := dA$ depends only on the class. The signature $(k, n - k)$ of g can be arbitrary. Conformally Einstein structures from ordinary conformal geometry are a special class of Weyl structures, corresponding to the choice of a closed – hence locally exact – 1-form A .

Inspired by Levi-Civita, Weyl established that to such a *Weyl structure* $(M, [(g, A)])$ is associated a *unique* connection D on TM satisfying:

- (A) D has no torsion;
- (B) $Dg = 2Ag$ for any representative (g, A) of the class $[(g, A)]$.

In any (local) coframe ω^μ , $\mu = 1, \dots, n$, for the cotangent bundle T^*M in which $g = g_{\mu\nu}\omega^\mu\omega^\nu$, the connection 1-forms $\Gamma^\mu{}_\nu$ of D , or equivalently the $\Gamma_{\mu\nu} := g_{\mu\rho}\Gamma^\rho{}_\nu$, are indeed uniquely defined from the more explicit conditions:

- (A') $d\omega^\mu + \Gamma^\mu{}_\nu \wedge \omega^\nu = 0$;
- (B') $Dg_{\mu\nu} := dg_{\mu\nu} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu} = 2Ag_{\mu\nu}$.

Then the *curvature* of this Weyl connection identifies with the collection of n^2 *curvature 2-forms*

$$\Omega^\mu{}_\nu := d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu,$$

which produce the *curvature tensor* $R^\mu{}_{\nu\rho\sigma}$ by expanding in the given coframe ω^μ

$$\Omega^\mu{}_\nu = \frac{1}{2}R^\mu{}_{\nu\rho\sigma}\omega^\rho \wedge \omega^\sigma.$$

It turns out that $R^\mu{}_{\nu\rho\sigma}$ is a *tensor density*, which means in particular that its vanishing is independent of the choice of a representative (g, A) , and hence as such, serves as a starting point for all invariants of a Weyl geometry $(M, [(g, A)])$, produced by covariant differentiation.

Other invariant objects are:

- the (Weyl-)Ricci tensor $R_{\mu\nu} := R^\rho{}_{\mu\rho\nu}$;
- its symmetric part $R_{(\mu\nu)} := \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu})$;
- its antisymmetric part $R_{[\mu\nu]} := \frac{1}{2}(R_{\mu\nu} - R_{\nu\mu})$.

In particular, an appropriately contracted Bianchi identity shows that in 3-dimensions

$$R_{[\mu\nu]} = -\frac{3}{2}F_{\mu\nu},$$

where $F = dA =: \frac{1}{2}F_{\mu\nu}\omega^\mu \wedge \omega^\nu$.

In [3], Élie Cartan proposed dynamical Einstein equations for a Weyl geometry $(M, [(g, A)])$ postulating that the trace-free part of the symmetric Ricci tensor vanishes

$$R_{(\mu\nu)} - \frac{1}{n}Rg_{\mu\nu} = 0, \tag{2.1}$$

where $R := g^{\mu\nu}R_{\mu\nu}$, with $g^{\mu\rho}g_{\rho\nu} = \delta^\mu{}_\nu$ and $n = \dim M$.

These equations (2.1) are called *Einstein–Weyl equations*, and a Weyl geometry satisfying (2.1) is called an *Einstein–Weyl structure*. The reason for this name is as follows.

Since a Weyl structure $(M, [g, A])$ with vanishing $F = dA \equiv 0$ is equivalent to a plain (pseudo-)conformal structure $(M, [g])$ and since the Weyl connection D then reduces to the Levi-Civita connection, these equations (2.1) are a natural generalization of Einstein’s field equations. According to Weyl’s approach, a gravity potential g is thereby *coupled* with an electromagnetic field $F = dA$.

3 Cartan's solution to the Einstein–Weyl vacuum equations

In [2], Cartan gave a geometric description of all solutions to the Einstein–Weyl equations (2.1) in 3-dimensions. In particular, he showed that there is a *one-to-one correspondence* between 3rd-order ODEs $y''' = H(x, y, y', y'')$ considered modulo point transformations of variables which satisfy certain two point-invariant conditions

$$W(H) \equiv 0, \quad (\text{Wünschmann})$$

$$C(H) \equiv 0, \quad (\text{Cartan})$$

and 3-dimensional Einstein–Weyl structures with *Lorentzian* metrics g of signature $(2, 1)$. Abbreviating $p := y'$, $q := y''$, in terms of the total differentiation operator

$$D := \partial_x + p\partial_y + q\partial_p + H\partial_q,$$

their explicit expressions are

$$W := 9DDH_q - 27DH_p - 18H_qDH_q + 18H_qH_p + 4H_q^3 + 54H_y, \quad (3.1)$$

$$C := 18H_{qq}DH_q - 12H_{qq}H_q^2 - 54H_{qq}H_p + 36H_{pq}H_q - 108H_{yq} + 54H_{pp}. \quad (3.2)$$

Although Cartan's geometric arguments [3] offer, in the Lorentzian setting, a complete – but abstract – understanding of the space of all solutions of the Einstein–Weyl equations (2.1), it is quite difficult to find *explicit* solutions to the Wünschmann–Cartan equations $0 \equiv W(H) \equiv C(H)$, which would provide workable formulas for such Einstein–Weyl structures.

Some particular solutions are known, e.g.,

$$H = \frac{3q^2}{2p}, \quad H = \frac{3q^2p}{p^2 + 1}, \quad H = q^{3/2}, \quad H = \alpha \frac{(2qy - p^2)^{3/2}}{y^2}, \quad \alpha \in \mathbb{R},$$

or the ‘horrible’

$$H = \frac{pq(-12 + 3pq - 8\sqrt{1 - pq}) + 8(1 + \sqrt{1 - pq})}{p^3}.$$

They were all obtained by rather *ad hoc* methods.

In fact, the main difficulty in getting a systematic approach to finding the solutions is an annoying nonlinearity of the Wünschmann condition $W \equiv 0$.

4 Third-order ODEs modulo point transformations of variables

It was Cartan [2] who solved the equivalence problem for 3rd order ODEs considered modulo point transformations. Nowadays, the result may be stated more elegantly in terms of a certain Cartan connection [7, 8], as follows.

To any 3rd order ODE

$$y''' = H(x, y, y', y''), \quad (4.1)$$

one associates a contact-like coframe on the space $J_4 \ni (x, y, p, q)$ of 2-jets of graphs $x \mapsto y(x)$:

$$\omega^1 := dy - p dx, \quad \omega^2 := dx, \quad \omega^3 := dp - q dx, \quad \omega^4 := dq - H(x, y, p, q) dx. \quad (4.2)$$

It follows that if a 3rd order ODE (4.1) undergoes a point transformation of variables

$$(x, y) \mapsto (\bar{x}, \bar{y}) = (\bar{x}(x, y), \bar{y}(x, y)),$$

then the 1-forms $(\omega^1, \omega^2, \omega^3, \omega^4)$ transform as

$$\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} \mapsto \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ u_4 & 0 & u_5 & 0 \\ u_6 & 0 & u_7 & u_8 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} =: \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \end{pmatrix}, \quad (4.3)$$

where the u_i are certain functions on J_4 .

Actually, Cartan assures us that the entire equivalence problem for 3rd order ODEs considered modulo point transformations of variables is the same as the equivalence problem for 1-forms (4.2), considered modulo transformations (4.3). There is a unique way of reducing these eight group parameters u_i to only three u_3, u_5, u_7 , the other ones being expressed in terms of them. This is achieved by forcing the exterior differentials of the θ^μ 's to satisfy the EDS (4.4) below.

Theorem 4.1 ([2, 7, 8]). *A 3rd order ODE $y''' = H(x, y, y', y'')$ with its associated 1-forms*

$$\omega^1 = dy - p dx, \quad \omega^2 = dx, \quad \omega^3 = dp - q dx, \quad \omega^4 = dq - H(x, y, p, q) dx,$$

uniquely defines a 7-dimensional fiber bundle $P_7 \rightarrow J_4$ over the space of second jets $J_4 \ni (x, y, p, q)$ and a unique coframe $\{\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3\}$ on P_7 enjoying structure equations of the shape

$$\begin{aligned} d\theta^1 &= \Omega_1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= (\Omega_1 - \Omega_3) \wedge \theta^2 + \boxed{B_1} \theta^1 \wedge \theta^3 - B_2 \theta^1 \wedge \theta^4, \\ d\theta^3 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^3 + \theta^2 \wedge \theta^4, \\ d\theta^4 &= (2\Omega_3 - \Omega_1) \wedge \theta^4 - \Omega_2 \wedge \theta^3 - \boxed{A_1} \theta^1 \wedge \theta^2, \\ d\Omega_1 &= \Omega_2 \wedge \theta^2 + (A_2 - 2C_1) \theta^1 \wedge \theta^2 + (3B_3 + E_1) \theta^1 \wedge \theta^3 + (2B_1 - 3B_4) \theta^1 \wedge \theta^4 + B_2 \theta^3 \wedge \theta^4, \\ d\Omega_2 &= \Omega_2 \wedge (\Omega_1 - \Omega_3) - A_3 \theta^1 \wedge \theta^2 + E_2 \theta^1 \wedge \theta^3 - (B_3 + E_1) \theta^1 \wedge \theta^4 + \boxed{C_1} \theta^2 \wedge \theta^3 \\ &\quad + (B_1 - 2B_4) \theta^3 \wedge \theta^4, \\ d\Omega_3 &= -C_1 \theta^1 \wedge \theta^2 + (2B_3 + E_1) \theta^1 \wedge \theta^3 + 2(B_1 - B_4) \theta^1 \wedge \theta^4 + 2B_2 \theta^3 \wedge \theta^4. \end{aligned} \quad (4.4)$$

Moreover, two equations $y''' = H(x, y, y', y'')$ and $\bar{y}''' = \bar{H}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'')$ are locally point equivalent if and only if there exists a local bundle isomorphism $\Phi: P_7 \xrightarrow{\sim} \bar{P}_7$ between the corresponding bundles $P_7 \rightarrow J_4$ and $\bar{P}_7 \rightarrow \bar{J}_4$ satisfying

$$\Phi^* \bar{\theta}^\mu = \theta^\mu \quad \text{and} \quad \Phi^* \bar{\Omega}_i = \Omega_i, \quad \mu = 1, 2, 3, 4, \quad i = 1, 2, 3.$$

Exactly 3 (boxed) invariants are primary: A_1, B_1, C_1 , while others express in terms of them and their covariant derivatives. Point equivalence to $\bar{y}''' = 0$ is characterized by $0 \equiv A_1 \equiv B_1 \equiv C_1$. Two relevant explicit expressions are

$$A_1 = \frac{1}{54} \frac{u_3^3}{u_1^3} W, \quad (\text{W in (3.1)})$$

$$C_1 = \frac{1}{54} \frac{u_3}{u_1^2} \left(C + \frac{1}{27} W_q \right). \quad (\text{C in (3.2)})$$

The seven 1-forms $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3)$ set up a Cartan connection $\hat{\omega}$ on P_7 via

$$\hat{\omega} := \begin{pmatrix} \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 & 0 & 0 \\ -\theta^2 & \Omega_3 - \frac{1}{2}\Omega_1 & 0 & 0 \\ \theta^3 & -\theta^4 & \frac{1}{2}\Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_2 \\ 2\theta^1 & \theta^3 & \theta^2 & -\frac{1}{2}\Omega_1 \end{pmatrix},$$

and the structure equations (4.4) are just the equations for the curvature \widehat{K} of this connection

$$d\widehat{\omega} + \widehat{\omega} \wedge \widehat{\omega} =: \widehat{K}.$$

Now, the structure equations (4.4) guarantee that the bundle P_7 is foliated by a 4-dimensional distribution annihilating the three 1-forms $(\theta^1, \theta^3, \theta^4)$, and that the leaf space M_3 of this foliation is equipped with a natural Weyl geometry, if and only if two among three primary invariants vanish identically

$$0 \equiv A_1(H) \equiv C_1(H).$$

A representative (g, A) of the concerned Weyl class $[(g, A)]$ on M_3 has then the signature $(2, 1)$ symmetric bilinear form

$$g := \theta^3\theta^3 + \theta^1\theta^4 + \theta^4\theta^1,$$

which is obtained as the determinant of the lower-left 2×2 submatrix of the connection matrix $\widehat{\omega}$, while the 1-form is defined as

$$A := \Omega_3.$$

It is thanks to the hypothesis $A_1 \equiv 0 \equiv C_1$ that g and A , originally defined on P_7 , descend on M_3 .

Furthermore, according to a result of Cartan in [3], any such Weyl geometry $[(g, A)]$ defined on such a leaf space M_3 is *automatically Einstein–Weyl*!

We stress that given $H = H(x, y, p, q)$ satisfying $A_1 \equiv 0 \equiv C_1$, or equivalently

$$W(H) \equiv 0 \equiv C(H),$$

one can in principle set up *explicit* formulas for the corresponding forms $\theta^1, \theta^3, \theta^4, \Omega_3$ on P_7 , and this in turn can provide *explicit formulas* for (g, A) on M_3 . However, one substantial obstacle is

Question 4.2. How to solve $W(H) \equiv 0 \equiv C(H)$?

5 PDE on the plane $z_y = F(x, y, z, z_x)$ modulo point transformations

We recall that in [9], it was shown that the equivalence problem for 3rd-order ODEs considered modulo point transformations of variables is in one-to-one correspondence with the equivalence problem for 4-dimensional para-CR structures of type $(1, 1, 2)$, cf. also [14, 15]. This thus suggests a new approach for constructing Lorentzian Einstein–Weyl structures via para-CR structures of type $(1, 1, 2)$. Instead of working with general para-CR structures of type $(1, 1, 2)$, we will concentrate on a subclass determined in the following way.

We start with a class of PDEs of the form

$$z_y = F(x, y, z, z_x),$$

considered modulo point transformations, for an unknown function $z = z(x, y)$. We then ask when this class defines a para-CR structure of type $(1, 1, 2)$.

To answer this (in Proposition 5.1), we need a little preparation. Using the abbreviation $z_x =: p$, we indeed consider such PDEs modulo point transformations of variables

$$(x, y, z) \longmapsto (\bar{x}, \bar{y}, \bar{z}) = (\bar{x}(x, y, z), \bar{y}(x, y, z), \bar{z}(x, y, z)).$$

This leads to an equivalence problem for the four 1-forms

$$\omega_0^1 := dz - p dx - F(x, y, z, p) dy, \quad \omega_0^2 := dp, \quad \omega_0^3 := dx, \quad \omega_0^4 := dy,$$

given up to transformations

$$\begin{pmatrix} \omega_0^1 \\ \omega_0^2 \\ \omega_0^3 \\ \omega_0^4 \end{pmatrix} \mapsto \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ u_4 & 0 & u_5 & u_6 \\ u_7 & 0 & u_8 & u_9 \end{pmatrix} \begin{pmatrix} \omega_0^1 \\ \omega_0^2 \\ \omega_0^3 \\ \omega_0^4 \end{pmatrix}. \quad (5.1)$$

Within this coframe $\{\omega_0^1, \omega_0^2, \omega_0^3, \omega_0^4\}$, in terms of the two operators

$$D := \partial_x + p\partial_z \quad \text{and} \quad \Delta := \partial_y + F\partial_z,$$

the exterior differential of any function $F = F(x, y, z, p)$ can be rewritten as

$$dF = F_z \omega_0^1 + F_p \omega_0^2 + DF \omega_0^3 + \Delta F \omega_0^4.$$

Proposition 5.1. *The coframe of 1-forms $\{\omega_0^1, \omega_0^2, \omega_0^3, \omega_0^4\}$ modulo transformations (5.1) defines a para-CR structure of type $(1, 1, 2)$ if and only if*

$$0 \equiv DF = F_x + pF_z.$$

Proof. The only nontrivial integrability condition required to constitute a true para-CR structure comes from

$$0 = d\omega_0^1 \wedge \omega_0^1 \wedge \omega_0^2 = -DF \omega_0^1 \wedge \omega_0^2 \wedge \omega_0^3 \wedge \omega_0^4. \quad \blacksquare$$

We will now show that for this class of para-CR structures there is an amazing coincidence between its main invariant, which will happen to be the Monge invariant with respect to p , and the classical Wünschmann invariant of the corresponding class of 3rd order ODEs modulo point transformations.

From now on, we will only consider PDEs $z_y = F(x, y, z, z_x)$ satisfying $DF \equiv 0$. Furthermore, we will also assume that another point-invariant condition holds

$$0 \neq F_{pp} \quad (\text{everywhere}).$$

Cartan's process leads one to choose more convenient representatives of these forms

$$\begin{aligned} \omega^1 &:= \omega_0^1, \\ \omega^2 &:= \omega_0^2 - \frac{\Delta F_{ppp} F_{pp} - \Delta F_{pp} F_{ppp} + 3F_p F_{pp} F_{zpp} - 3F_{pp}^2 F_{zp} - 2F_p F_{ppp} F_{zp}}{6F_{pp}^3} \omega_0^1, \\ \omega^3 &:= \omega_0^3 + F_p \omega_0^4 - \frac{1}{3} \frac{F_{ppp}}{F_{pp}} \omega_0^1, \\ \omega^4 &:= F_{pp} \omega_0^4 + \frac{4F_{pp}^2 - 3F_{pp} F_{pppp}}{18F_{pp}^2} \omega_0^1, \end{aligned}$$

and we will use this choice in the sequel.

Using Cartan's method, it is then straightforward to solve the equivalence problem for point equivalence classes of such PDEs $z_y = F(x, y, z, z_x)$. The solution is summarized in the following

Theorem 5.2. *A PDE system $z_y = F(x, y, z, z_x)$ satisfying the two point-invariant conditions*

$$DF \equiv 0 \neq F_{z_x z_x},$$

with its associated 1-forms $\omega^1, \omega^2, \omega^3, \omega^4$ as above, uniquely defines a 7-dimensional principal H_3 -bundle $H_3 \rightarrow P_7 \rightarrow J_4$ over the space of first jets $J_4 \ni (x, y, z, p)$ with the (reduced) structure group H_3 consisting of matrices

$$\begin{pmatrix} u_3 u_5 & 0 & 0 & 0 \\ 0 & u_3 & 0 & 0 \\ -u_3 u_8 & 0 & u_5 & 0 \\ -\frac{u_3 u_8^2}{2u_5} & 0 & u_8 & \frac{u_5}{u_3} \end{pmatrix}, \quad u_3 \in \mathbb{R}^*, \quad u_5 \in \mathbb{R}^*, \quad u_8 \in \mathbb{R},$$

together with a unique coframe $\{\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3\}$ on P_7 where

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \end{pmatrix} := \begin{pmatrix} u_3 u_5 & 0 & 0 & 0 \\ 0 & u_3 & 0 & 0 \\ -u_3 u_8 & 0 & u_5 & 0 \\ -\frac{u_3 u_8^2}{2u_5} & 0 & u_8 & \frac{u_5}{u_3} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix},$$

such that the coframe enjoys precisely the structure equations (4.4). This time however, the curvature invariants $A_1, A_2, A_3, B_1, B_2, B_3, B_4, C_1, C_2, C_3, E_1, E_2$ depend on $F = F(x, y, z, p)$ and its derivatives up to order 6.

Two relevant explicit expressions are

$$A_1 = -\frac{1}{54} \frac{1}{u_3^3} \frac{M}{F_{pp}^3}, \quad C_1 = \frac{1}{3} \frac{1}{u_3^2 u_5} \frac{K}{F_{pp}^5},$$

where

$$\begin{aligned} M &:= 9F_{ppppp}F_{pp}^2 - 45F_{pppp}F_{ppp}F_{pp} + 40F_{ppp}^3, \\ K &:= \Delta F_{ppppp}F_{pp}^3 - 5\Delta F_{pppp}F_{pp}^2F_{ppp} + 12\Delta F_{ppp}F_{pp}F_{ppp}^2 - 12\Delta F_{pp}F_{ppp}^3 - 4\Delta F_{ppp}F_{pp}^2F_{pppp} \\ &\quad + 9\Delta F_{pp}F_{pp}F_{ppp}F_{pppp} - \Delta F_{pp}F_{pp}^2F_{ppppp} + 5F_pF_{pp}^3F_{ppppz} + 6F_{pp}^4F_{pppz} \\ &\quad - 20F_pF_{pp}^2F_{ppp}F_{pppz} - 12F_{pp}^3F_{ppp}F_{ppz} + 36F_pF_{pp}F_{ppp}^2F_{ppz} - 12F_pF_{pp}^2F_{pppp}F_{ppz} \\ &\quad + 8F_{pp}^2F_{ppp}^2F_{pz} - 24F_pF_{ppp}^3F_{pz} - 3F_{pp}^3F_{pppp}F_{pz} + 18F_pF_{pp}F_{ppp}F_{pppp}F_{pz} \\ &\quad - 2F_pF_{pp}^2F_{pppp}F_{pz}. \end{aligned}$$

Two equations $z_y = F(x, y, z, z_x)$ and $\bar{z}_{\bar{y}} = \bar{F}(\bar{x}, \bar{y}, \bar{z}, \bar{z}_{\bar{x}})$ satisfying $DF = 0 \neq F_{z_x z_x}$ and $\overline{DF} \equiv 0 \neq \bar{F}_{\bar{z}_{\bar{x}} \bar{z}_{\bar{x}}}$ are locally point equivalent if and only if there exists a bundle isomorphism $\Phi: P_7 \xrightarrow{\sim} \bar{P}_7$ between the corresponding principal bundles $H_3 \rightarrow P_7 \rightarrow J_4$ and $\bar{H}_3 \rightarrow \bar{P}_7 \rightarrow \bar{J}_4$ satisfying

$$\Phi^* \bar{\theta}^\mu = \theta^\mu \quad \text{and} \quad \Phi^* \bar{\Omega}_i = \Omega_i, \quad \mu = 1, 2, 3, 4, \quad i = 1, 2, 3.$$

This theorem enables one to think about the geometry of a PDE $z_y = F(x, y, z, z_x)$ with $DF \equiv 0 \neq F_{z_x z_x}$, considered modulo point transformations of variables, as the geometry of a certain 3rd order ODE $y''' = H(x, y, y', y'')$, also considered modulo point transformations. In particular, one can ask how big is the subclass of point nonequivalent 3rd order ODEs which are related to PDEs $z_y = F(x, y, z, z_x)$ with $DF \equiv 0 \neq F_{z_x z_x}$.

We will not answer this question in this paper. Instead, we concentrate on the Einstein–Weyl geometric aspect of the above observation.

Since the EDS staying behind the PDEs $z_y = F(x, y, z, z_x)$ with $DF \equiv 0 \neq F_{z_x z_x}$ is visibly the same as the EDS for 3rd order ODEs $y''' = H(x, y, y', y'')$, one can look for PDEs $z_y = F(x, y, z, z_x)$ with $DF \equiv 0 \neq F_{z_x z_x}$, which in addition satisfy $A_1 = C_1 = 0$, and build a corresponding Einstein Weyl geometry, not in terms of $H(x, y, y', y'')$ satisfying $W(H) \equiv C(H) \equiv 0$, but in terms of the function $F(x, y, z, z_x)$ satisfying $DF \equiv M(F) \equiv K(F) \equiv 0$. If only $M(F) \equiv 0$, there exists a conformal Lorentzian metric on the leaf space of the integrable distribution in P_7 annihilated by $\{\theta^1, \theta^3, \theta^4\}$, and when moreover $K(F) \equiv 0$, all this produces Einstein–Weyl geometries. Actually, we gain the following

Theorem 5.3. *A PDE $z_y = F(x, y, z, z_x)$ with $DF \equiv 0 \neq F_{z_x z_x}$ defines a bilinear form \tilde{g} of signature $(+, +, -, 0, 0, 0, 0)$ on the bundle $P_7 \ni (x, y, z, p, u_3, u_5, u_8)$:*

$$\begin{aligned} \tilde{g} = & \theta^3 \theta^3 + \theta^1 \theta^4 + \theta^4 \theta^1 = \frac{u_5^2}{9F_{pp}^2} \left\{ (3F_{pp}[dx + F_p dy] - F_{ppp}[dz - p dx - F dy])^2 \right. \\ & \left. + (dz - p dx - F dy)(18F_{pp}^3 dy + [4F_{ppp}^2 - 3F_{pp}F_{pppp}][dz - p dx - F dy]) \right\}, \end{aligned}$$

degenerate along the rank 4 integrable distribution \mathcal{D}_4 which is the annihilator of $\theta^1, \theta^3, \theta^4$.

The PDE $z_y = F(x, y, z, z_x)$ with $DF \equiv 0 \neq F_{z_x z_x}$ also defines the 1-form

$$\Omega_3 := r_x dx + r_y dy + r_z dz + \frac{1}{3} d[\log(u_5^3 F_{pp})],$$

where

$$\begin{aligned} r_x = & \frac{1}{3F_{pp}^4} \left\{ \Delta F_{ppp} F_{pp}^2 - \Delta F_{pp} F_{pp} F_{ppp} + 3F_p F_{pp}^2 F_{ppz} - F_{pp}^3 F_{pz} - 2F_p F_{pp} F_{ppp} F_{pz} \right. \\ & - \Delta F_{pppp} F_{pp}^2 p + 3\Delta F_{ppp} F_{pp} F_{ppp} p - 3\Delta F_{pp} F_{ppp}^2 p + \Delta F_{pp} F_{pp} F_{pppp} p \\ & - 4F_p F_{pp}^2 F_{pppz} p - 2F_{pp}^3 F_{ppz} p + 9F_p F_{pp} F_{ppp} F_{ppz} p + F_{pp}^2 F_{ppp} F_{pz} p \\ & \left. - 6F_p F_{ppp}^2 F_{pz} p + 2F_p F_{pp} F_{pppp} F_{pz} p \right\}, \\ r_y = & \frac{1}{3F_{pp}^4} \left\{ -\Delta F_{pppp} F F_{pp}^2 + \Delta F_{ppp} F_p F_{pp}^2 - \Delta F_{pp} F_{pp}^3 + 3\Delta F_{ppp} F F_{pp} F_{ppp} \right. \\ & - \Delta F_{pp} F_p F_{pp} F_{ppp} - 3\Delta F_{pp} F F_{ppp}^2 + \Delta F_{pp} F F_{pp} F_{pppp} - 4F F_p F_{pp}^2 F_{pppz} \\ & + 3F_p^2 F_{pp}^2 F_{ppz} - 2F F_{pp}^3 F_{ppz} + 9F F_p F_{pp} F_{ppp} F_{ppz} - 3F_p F_{pp}^3 F_{pz} \\ & - 2F_p^2 F_{pp} F_{ppp} F_{pz} + F F_{pp}^2 F_{ppp} F_{pz} - 6F F_p F_{ppp}^2 F_{pz} + 2F F_p F_{pp} F_{pppp} F_{pz} + 3F_{pp}^4 F_z \left. \right\}, \\ r_z = & \frac{1}{3F_{pp}^4} \left\{ \Delta F_{pppp} F_{pp}^2 - 3\Delta F_{ppp} F_{pp} F_{ppp} + 3\Delta F_{pp} F_{ppp}^2 - \Delta F_{pp} F_{pp} F_{pppp} + 4F_p F_{pp}^2 F_{pppz} \right. \\ & \left. + 2F_{pp}^3 F_{ppz} - 9F_p F_{pp} F_{ppp} F_{ppz} - F_{pp}^2 F_{ppp} F_{pz} + 6F_p F_{ppp}^2 F_{pz} - 2F_p F_{pp} F_{pppp} F_{pz} \right\}. \end{aligned}$$

The degenerate bilinear form \tilde{g} descends to a Lorentzian conformal class $[g]$ on the leaf space M_3 of the distribution \mathcal{D}_4 , if and only if the Monge invariant $M(F) \equiv 0$ vanishes identically.

When $M(F) \equiv 0$, the local coordinates on M_3 are (x, y, z) with the projection

$$\begin{aligned} P_7 & \longrightarrow M_3, \\ (x, y, z, p, u_3, u_5, u_8) & \longmapsto (x, y, z), \end{aligned}$$

and the conformal class $[g]$ has a representative which is explicitly expressed in terms of dx, dy, dz , with coefficients depending only on (x, y, z) .

Next, Ω_3 descends to a 1-form denoted A given up to the differential of a function on $M_3 \ni (x, y, z)$, if and only if $\mathbf{K}(F) \equiv 0$.

Moreover, the pair (\tilde{g}, Ω_3) descends to a representative of a Weyl structure $[(g, A)]$ on M_3 , if and only if both $\mathbf{M}(F) \equiv 0$ and $\mathbf{K}(F) \equiv 0$.

Finally, this Weyl structure is actually Einstein–Weyl, namely it satisfies (2.1).

6 Transformation of the Wünschmann invariant into the Monge invariant

As we now know, PDEs $z_y = F(x, y, z, z_x)$ with $DF \equiv 0 \neq F_{z_x z_x}$ satisfying $\mathbf{A}_1 \equiv 0 \equiv \mathbf{C}_1$ always define an Einstein–Weyl geometry on the leaf space M_3 of the integrable distribution in P_7 annihilated by $\{\theta^1, \theta^3, \theta^4\}$.

The advantage of looking at a Weyl geometry from the PDE $z_y = F(x, y, z, z_x)$ point of view rather than from the ODE side $y''' = H(x, y, y', y'')$, is that now the Wünschmann invariant of the ODE becomes the much simpler and classical *Monge invariant*

$$\mathbf{A}_1(H) \sim \mathbf{M}(F) = 9F_{pp}^2 F_{ppppp} - 45F_{pp} F_{pppp} F_{ppp} + 40F_{ppp}^3.$$

Serendipitously, the identical vanishing $\mathbf{M}(F) \equiv 0$ is well known to be equivalent to the condition that the graph of $p \mapsto F(p)$ is contained in a conic of the (p, F) -plane, with parameters (x, y, z) . More precisely,

$$0 \equiv \mathbf{M}(F) \iff AF^2 + 2BFp + Cp^2 + 2KF + 2Lp + M \equiv 0, \quad (6.1)$$

for some functions A, B, C, K, L, M depending only on (x, y, z) .

Thus, passing from the formulation of Einstein–Weyl’s equations in terms of a 3rd order ODE $y''' = H(x, y, y', y'')$ to the formulation in terms of a PDE $z_y = F(x, y, z, z_x)$, we are able to find a rather *large class of solutions* to the equation

$$\mathbf{W}(H) \equiv 0.$$

Indeed, by replacing $\mathbf{W}(H) \rightsquigarrow \mathbf{M}(F)$, the solution (6.1) is just conical!

7 How to construct new explicit Lorentzian Einstein–Weyl metrics?

But remember we also have to assure that

$$0 \equiv DF = \partial_x F + p \partial_z F.$$

The simultaneous conditions $DF \equiv 0 \equiv \mathbf{M}(F)$ can be achieved for instance by taking F satisfying

$$aF^2 + 2bF(z - px) + c(z - px)^2 + 2kF + 2l(z - px) + m \equiv 0,$$

with a, b, c, k, l, m being now functions of y *only*!

From now on, we will analyze this special solution for $\mathbf{M}(F) \equiv 0 \equiv DF$. The simplest case occurs when avoiding square root by choosing

$$a := 0,$$

so that

$$F := \frac{-c(z - xp)^2 - 2l(z - xp) - m}{2b(z - xp) + 2k}. \quad (7.1)$$

Here

$$b = b(y), \quad c = c(y), \quad k = k(y), \quad l = l(y), \quad m = m(y)$$

are free arbitrary differentiable functions of one variable y .

A direct check shows that *remarkably this solution (7.1) also satisfies $K(F) \equiv 0$!*

Proposition 7.1. *All such*

$$F := \frac{-c(z - xp)^2 - 2l(z - xp) - m}{2b(z - xp) + 2k}$$

with any functions b, c, k, l, m of y , lead to Einstein–Weyl structures in 3-dimensions.

Performing the Cartan procedure to determine the coframe $\{\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3\}$, projecting both $\theta^3\theta^3 + \theta^1\theta^4 + \theta^4\theta^1$ and Ω_3 to the leaf space of the annihilator M^3 of $\{\theta^1, \theta^3, \theta^4\}$, equipping $M_3 \equiv \mathbb{R}^3$ with coordinates (x, y, z) , we therefore obtain *functionally parameterized* Einstein–Weyl structures (g, A) on $\mathbb{R}^3 \ni (x, y, z)$ represented by the signature $(2, 1)$ Lorentzian metric

$$g := (k + bz)^2 dx^2 + x^2(l^2 - cm) dy^2 + x^2 b^2 dz^2 \\ + 2x(ckz - blz + kl - bm) dx dy - 2xb(k + bz) dx dz - 2x^2(ck - bl) dy dz,$$

together with the differential 1-form

$$A := \frac{-ck + bl + b'k - bk'}{x(ck^2 - 2bkl + b^2m)} (xb dz - (k + bz) dx) \\ + \frac{bl^2 - cbm - b'kl + bb'm + ckk' - bk'l}{ck^2 - 2bkl + b^2m} dy.$$

An independent direct check confirms that equations (1.1) are indeed identically fulfilled.

As regards the Cotton tensor, we compute its 5 components, and find that they are not identically zero. Hence the obtained Einstein–Weyl structures are generically conformally non-flat. Thus, Theorem 1.1 is established. The story for Theorem 1.4 is quite similar.

Next, without assuming $A \equiv 0$ in (6.1), let us now make the ansatz that

$$aF^2 + 2bF(z - xp) + c(z - xp)^2 + 2kF + 2l(z - xp) + m \equiv 0,$$

for some arbitrary functions a, b, c, k, l, m of y . The (two) solutions F automatically satisfy $DF \equiv 0 \equiv M(F)$.

Since the solutions to Monge's equation are conics in the (p, F) -plane, we can rewrite in a hyperbolic setting

$$(aF + b(z - xp) + c)^2 - (kF + l(z - xp) + m)^2 \equiv 1,$$

with changed functions a, b, c, k, l, m of y . To avoid transcendental functions in computations, we parametrize $\cosh t = \frac{1+q^2}{2q}$ and $\sinh t = \frac{1-q^2}{2q}$, and then, solving for F and for $z - xp$, we may start from

$$F = a(y) \frac{1+q^2}{2q} + b(y) \frac{1-q^2}{2q} + c(y), \\ z - xp = k(y) \frac{1+q^2}{2q} + l(y) \frac{1-q^2}{2q} + m(y),$$

again with (changed) free functions a, b, c, k, l, m of y . Taking

$$\omega_0^1 := d(z - xp) + x dp - F dy, \quad \omega_0^2 := dx, \quad \omega_0^3 := dy, \quad \omega_0^4 := dp,$$

and performing para-CR Cartan reduction to an $\{e\}$ -structure/connection, we obtain

Proposition 7.2. *The second invariant condition $K(F) \equiv 0$ holds precisely in the following two cases:*

- (1) $k = l$;
- (2) $c = m'$ and $a = \frac{bl + kk' - ll'}{k}$.

In case (1), we obtain Einstein–Weyl structures for all free functions a, b, c, l, m of y given by

$$g := 2\tau^1\tau^2 + (\tau^3)^2, \quad A := -\frac{2(a+b)}{x(a-b)l}\tau^2 - \frac{c-m'}{x(a-b)}\tau^3,$$

where

$$\tau^1 := x(a+b)dy - 2ldx, \quad \tau^2 := -\frac{1}{2}x(a-b)dy, \quad \tau^3 := xc dy - x dz + (z-m)dx.$$

We verify that these Einstein–Weyl structures have nontrivial $F = dA \neq 0$ and nontrivial $\text{Cotton}([g]) \neq 0$.

In case (2), we obtain Einstein–Weyl structures given by

$$g := 2\tau^1\tau^2 + (\tau^3)^2, \quad A := d[\log(x^2e)],$$

where

$$\begin{aligned} \tau^1 &:= (k+l)k dx + x(bk - bl + kk' - ll') dy, \\ \tau^2 &:= \frac{1}{2}(k-l)k dx + \frac{1}{2}x(bk - bl + kk' - ll') dy, \\ \tau^3 &:= -(z-m)k dx - xkm dy + xk dz. \end{aligned}$$

But this structure, which depends on 3 functions b, k, l of y , is flat

$$dA \equiv 0 \equiv \text{Cotton}([g]).$$

Finally, without replacing p by $z - xp$, let us make the ansatz that

$$aF^2 + 2bFp + cp^2 + 2kF + 2lp + m \equiv 0.$$

Dealing similarly with the hyperbolic case,

$$\begin{aligned} F &= a(y)\frac{1+q^2}{2q} + b(y)\frac{1-q^2}{2q} + c(y), \\ p &= k(y)\frac{1+q^2}{2q} + l(y)\frac{1-q^2}{2q} + m(y), \end{aligned}$$

we obtain nontrivial Einstein–Weyl structures. For instance, when $k = l$ as in (1) above

$$g := 2\tau^1\tau^2 + (\tau^3)^2, \quad A := -\frac{m'}{(a-b)l}\tau^3,$$

where

$$\tau^1 := 2ldx + (a+b)dy, \quad \tau^2 := -\frac{1}{2}(a-b)dy, \quad \tau^3 := dz - m dx + (a+b)dy.$$

Note that this is again nontrivial

$$dA \neq 0 \neq \text{Cotton}([g]).$$

and note that we do not have x, z dependence here.

8 Transforming $z_y = F(z_x)$ into $w''' = w''H(t)$

We end up by exploring a link between our PDE systems and 3rd order ODEs. For simplicity, we will assume that $F = F(z_x)$ depends only on $p = z_x$.

To avoid notational confusion, 3rd-order ODEs will now be denoted as $w''' = H(t, w, w', w'')$, and the fundamental 1-forms as

$$\begin{aligned}\omega^1 &:= dz - p dx - F(p) dy, & \theta^1 &:= dw - w_1 dt, \\ \omega^2 &:= dp, & \theta^2 &:= dt, \\ \omega^3 &:= dx, & \theta^3 &:= dw_1 - w_2 dt, \\ \omega^4 &:= dy, & \theta^4 &:= dw_2 - H(t, w, w_1, w_2) dt.\end{aligned}$$

We ask what equivalence class of 3rd-order ODE's corresponds to the equivalence class of PDEs $z_y = F(z_x)$, still with $F_{pp} \neq 0$, and under the G -structures of Sections 4 and 5.

For this, since ω^1 and θ^1 are both defined up to plain dilations $\omega^1 \sim u\omega^1$ and $\theta^1 \sim u\theta^1$, we transform ω^1 in order to make the shape of θ^1 appear, using that F depends only on p

$$\omega^1 = d(z - xp - yF(p)) - (-x - yF_p(p)) dp =: dw - w_1 dt$$

with $t := p$, $w := z - xp - yF(p)$, $w_1 := -x - yF_p(p)$. With this, $\omega^2 = dp = dt = \theta^2$. Next, using $\omega^3 \sim -\omega^3 - u dy$, it comes

$$\begin{aligned}\omega^3 &= dx = -[d(-x - yF_p(p)) + yF_{pp}(p) dp + F_p(p) dy] \\ &\sim [dw_1 + yF_{pp}(p) dp + F_p(p) dy] - \underline{F_p(p) dy} = dw_1 - (-yF_{pp}(p)) dp,\end{aligned}$$

whence $w_2 := -yF_{pp}(p)$.

A last computation using $\omega^4 \sim u\omega^4$

$$\omega^4 = dy = -\frac{1}{F_{pp}(p)}[d(-yF_{pp}(p)) + yF_{ppp}(p) dp] \sim dw_2 - (-yF_{ppp}(p) dp),$$

shows that the right-hand side function $H = H(t, w_2)$ of the associated ODE $w''' = H$ is independent of w, w_1 as it must be

$$H := -yF_{ppp}(p) = w_2 \frac{F_{ttt}(t)}{F_{tt}(t)}.$$

Hence

$$w''' = w'' \frac{F_{ttt}(t)}{F_{tt}(t)}$$

is the 3rd order ODE associated to the para-CR structure given by $z_y = F(z_x)$. Observe that $z_y = \frac{1}{2}z_x^2$ becomes $w''' = 0$, leading to the flat Einstein–Weyl structure.

Assertion 8.1. *The Wünschmann invariant for ODEs $w''' = w'' \frac{F_{ttt}(t)}{F_{tt}(t)}$, where $F(t)$ with $F_{tt} \neq 0$ is an arbitrary function of one variable, corresponds to the Monge invariant of the PDE $z_y = F(z_x)$:*

$$\text{Wünschmann} \left(w_2 \frac{F_{ttt}(t)}{F_{tt}(t)} \right) = \frac{9F_{tt}(t)F_{tttt}(t) - 45F_{tt}F_{ttt}F_{ttt} + 40F_{ttt}^3}{F_{tt}^3} = \frac{\text{Monge}(F)}{F_{tt}^3}.$$

Proof. Among the 25 terms of Wünschmann's invariant shown in the Introduction, only 7 remain thanks to $0 \equiv H_w \equiv H_{w_1}$:

$$\begin{aligned}\text{Wünschmann}(H(t, w_2)) &= -9HH_{w_2}H_{w_2w_2} - 9H_tH_{w_2w_2} + 18HH_{tw_2w_2} - 18H_{w_2}H_{tw_2} \\ &\quad + 9H_{ttw_2} + 4H_{w_2}^3 + 9H^2H_{w_2w_2w_2},\end{aligned}$$

and a direct substitution of $H := w_2 \frac{F_{ttt}(t)}{F_{tt}(t)}$ leads to the result. ■

Acknowledgements

Insights of the anonymous referees are gratefully acknowledged. This collaboration is supported by the National Science Center, Poland, grant number 2018/29/B/ST1/02583.

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