Degrees $d \geqslant \big(\sqrt{n}\log n\big)^n$ and $d \geqslant \big(n\log n\big)^n$ in the Conjectures of Green-Griffiths and of Kobayashi[1]

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ABSTRACT. Once first answers in any dimension to the Green-Griffiths and Kobayashi conjectures for generic algebraic hypersurfaces $\mathbb{X}^{n-1} \subset \mathbb{P}^n(\mathbb{C})$ have been reached, the principal goal is to decrease (to improve) the degree bounds, knowing that the 'celestial' horizon lies near $d \geqslant 2n$.

For Green-Griffiths algebraic degeneracy of entire holomorphic curves, we obtain:

$$d \ \geqslant \ \big(\sqrt{n}\log n\big)^n,$$

and for Kobayashi-hyperbolicity (constancy of entire curves), we obtain:

$$d \ \geqslant \ \big(n\log n\big)^n.$$

The latter improves $d \geqslant n^{2n}$ obtained by Merker in arxiv.org/1807/11309/.

Admitting a certain technical conjecture $I_0 \ \geqslant \ \widetilde{I}_0$, the method employed (Diverio-Merker-Rousseau, Bérczi, Darondeau) conducts to constant power n, namely to:

$$d \ \geqslant \ 2^{5n} \qquad \text{and, respectively, to:} \qquad d \ \geqslant \ 4^{5n}.$$

In Spring 2021, a forthcoming prepublication based on intensive computer explorations will present several subconjectures supporting the belief that $I_0 \geqslant \widetilde{I}_0$, a conjecture which will be established up to dimension $n = 50$.

Dedicated to Professor Nguyen Tu Cuong on the occasion of his 70th birthday

1. Introduction

The goal is to establish that generic algebraic hypersurfaces of the projective space satisfy the Green-Griffiths conjecture, as well as their complements, with improvements on lower degree bounds.

Theorem 1.1. *For a generic hypersurface $\mathbb{X}^{n-1} \subset \mathbb{P}^n(\mathbb{C})$ of degree:*

$$d \ \geqslant \ \big(\sqrt{n}\log n\big)^n \qquad (\forall\, n \geqslant \mathrm{N_{GG}}),$$

(1) *there exists a proper subvariety $\mathbb{Y} \subset \mathbb{P}^n$ of codimension $\geqslant 2$ such that all nonconstant entire holomorphic curves $f\colon \mathbb{C} \longrightarrow \mathbb{P}^n \backslash \mathbb{X}$ have image $f(\mathbb{C}) \subset \mathbb{Y}$ in fact contained in $\mathbb{Y}$;*

(2) *there exists a proper subvariety $\mathbb{W} \subset \mathbb{X}$ of codimension $\geqslant 2$ such that all nonconstant entire holomorphic curves $f\colon \mathbb{C} \longrightarrow \mathbb{X}$ have image $f(\mathbb{C}) \subset \mathbb{W}$ in fact contained in $\mathbb{W}$.*

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This lower degree bound:

$$d \geqslant \boldsymbol{d}_{\mathrm{GG}}(n) := \big(\sqrt{n}\log n\big)^n$$

improves $d \geqslant (5n)^2\, n^n$ of [6] and improves $d \geqslant 2^{n^5}$ of [9]. In the demonstrations, we will treat mainly the details of the complement case **(1)**, since the computations in the compact case **(2)** are essentially similar, thanks to Darondeau's works [4, 5, 6].

By [15], any solution to the Green-Griffiths conjecture in all dimensions n for hypersurfaces of degrees $d \geqslant \boldsymbol{d}_{\mathrm{GG}}(n)$ implies a solution to the Kobayashi conjecture in all dimensions n for hypersurfaces of degrees:

$$d \geqslant \boldsymbol{d}_{\mathrm{K}}(n) := \boldsymbol{d}_{\mathrm{GG}}(2n).$$

Rounding off a small technical improvement of Theorem 1.1 in order to present only an elegant degree bound, we obtain as a corollary the following

Theorem 1.2. *For a generic hypersurface $\mathbb{X}^{n-1} \subset \mathbb{P}^n(\mathbb{C})$ of degree:*

$$d \geqslant \big(n\log n\big)^n \qquad (\forall\, n \geqslant \mathrm{N_K})$$

(1) *$\mathbb{P}^n \big\backslash \mathbb{X}^{n-1}$ is Kobayashi-hyperbolically imbedded in $\mathbb{P}^n$;*

(2) *$\mathbb{X}^{n-1}$ is Kobayashi-hyperbolic.*

An inspection of the end of Section **10** shows that the dimensions $\mathrm{N_{GG}}$ and $\mathrm{N_K}$ at which these statements begin to hold true can be made effective.

Theorem 1.2 improves the degree bound $d \geqslant n^{2n}$ obtained in [13]. For standard presentations of the research field, and for up-to-date history, including degree bound comparisons, the reader is referred to the introductions of the articles [13, 15, 3, 7, 11, 8, 2, 6, 5, 4, 16, 1, 9], listed in chronological order of prepublication.

Under the technical assumption (or conjecture):

$$I_0 \geqslant \widetilde{I}_0,$$

the explanation of which the reader will find in Section **2**, and which is equivalent to Problem 4.3, we obtain better results.

Theorem 1.3. *If $I_0 \geqslant \widetilde{I}_0$ holds true, then for a generic hypersurface $\mathbb{X}^{n-1} \subset \mathbb{P}^n(\mathbb{C})$ of degree:*

$$d \geqslant 2^{5n} \qquad (\forall\, n \geqslant 10),$$

the two conclusions **(1)** *and* **(2)** *of Theorem 1.1 hold true.*

Similarly, we also obtain as corollary the

Theorem 1.4. *Under the same technical assumption $I_0 \geqslant \widetilde{I}_0$, the conclusions* **(1)** *and* **(2)** *of Theorem 1.2 hold true in degree:*

$$d \geqslant 4^{5n} \qquad (\forall\, n \geqslant 20).$$

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Damian Brotbek, who first proved the *long-standing* Kobayashi hyperbolicity conjecture — exactly as Kobayashi himself originally stated it in 1970 [birth year of the second author] — with fully rigorous arguments, for generic algebraic hypersurfaces of arbitrary dimension, evaluated the Ph.D. of the second author, defended on September 30$^{\text{th}}$, 2020, at the Orsay Mathematics Institute, Paris-Saclay University.

2. Preliminary: Link with Darondeau's Work

This section continues [6], and goes slightly beyond. The jet order $\kappa = n$ will be chosen equal to the dimension n, because some reflections on the concerned estimates convince that any choice of $\kappa > n$ cannot improve the degree bound anyway.

Let $n \geqslant 1$ be an integer. Let $t_1, \ldots, t_n$ be formal variables. Introduce:

$$C(t_1, \ldots, t_n) \,:=\, \prod_{1 \leqslant i < j \leqslant n} \frac{t_j - t_i}{t_j - 2\, t_i} \prod_{2 \leqslant i < j \leqslant n} \frac{t_j - 2\, t_i}{t_j - 2\, t_i + t_{i+1}}.$$

As explained in [6], this rational expression possesses an iterated Laurent series at the origin as:

$$C(t) \,=\, \sum_{\substack{k_1, \ldots, k_n \,\in\, \mathbb{Z} \\ k_1 + \cdots + k_n \,=\, 0}} C_{k_1, \ldots, k_n}\, t_1^{k_1} \cdots t_n^{k_n},$$

for certain coefficients $C_{k_1, \ldots, k_n}$; soon, this object $C(t_1, t_2, \ldots, t_n)$ will be re-interpreted as a standard converging power series $C(w_2, \ldots, w_n)$ in terms of alternative new variables $(w_2, \ldots, w_n)$, hence it is not necessary to recall what an iterated Laurent series is.

For certain integer weights $a_1, \ldots, a_n \in \mathbb{N}^*$, introduce also an expression which comes from an application of the so-called *holomorphic Morse inequalities:*

$$f_0(t) \,:=\, \big(a_1 t_1 + \cdots + a_n t_n\big)^{n^2}.$$

It expands:

$$f_0(t) \,=\, \sum_{\substack{m_1, \ldots, m_n \geqslant 0 \\ m_1 + \cdots + m_n \,=\, n^2}} \frac{(n^2)!}{m_1! \cdots m_n!} \big(a_1\, t_1\big)^{m_1} \cdots \big(a_n\, t_n\big)^{m_n},$$

by means of (integer) multinomial coefficients:

$$M_{m_1, \ldots, m_n} \,:=\, \frac{(n^2)!}{m_1! \cdots m_n!}.$$

It is well known that the binomial $\binom{2n}{n}$ is the unique largest one among all the $\binom{2n}{i}$ with $0 \leqslant i \leqslant 2n$. In fact, an application of Stirling's asymptotic formula:

$$n! \underset{n \to \infty}{\sim} \sqrt{2\pi\, n} \Big(\frac{n}{e}\Big)^n \left[1 + \frac{1}{12\, n} + \frac{1}{288\, n^2} - \frac{139}{51\,840\, n^3} - \frac{571}{2\,488\,320\, n^4} + \mathrm{O}\big(\frac{1}{n^5}\big)\right],$$

shows that asymptotically as $n \longrightarrow \infty$:

$$\binom{2\, n}{n} \;\sim\; \frac{2^{2\, n}}{\sqrt{\pi\, n}} \left[1 - \frac{1}{8\, n} + \frac{1}{128\, n^2} + \frac{5}{1024\, n^3} - \frac{21}{32\,768\, n^4} + \mathrm{O}\Big(\frac{1}{n^5}\Big)\right].$$

Similarly, the *central multinomial coefficient:*

$$M_{n, \ldots, n} \,:=\, \frac{(n^2)!}{n! \cdots n!} \,=\, \frac{(n^2)!}{(n!)^n},$$

happens to be the unique largest one, as states the next observation (*see* also Lemma 5.1).

Lemma 2.1. *For all integers $m_1, \ldots, m_n \geqslant 0$ with $m_1 + \cdots + m_n = n^2$ and $(m_1, \ldots, m_n) \neq (n, \ldots, n)$, the corresponding multinomial coefficients are smaller than the central one:*

$$M_{m_1, \ldots, m_n} \,<\, M_{n, \ldots, n}.$$

Proof. This amounts to verify that:

$$\frac{n!}{m_1!} \cdots \frac{n!}{m_i!} \cdots \frac{n!}{m_n!} \overset{?}{<} 1.$$

The $m_i = n$ are neutral, for $\frac{n!}{n!} = 1$. By assumption, at least one $m_i \neq n$.

- When $m_i < n$, simplify:

$$\frac{n!}{m_i!} = n\,(n-1)\cdots\big(m_i+1\big).$$

- When $m_i > n$, simplify:

$$\frac{n!}{m_i!} = \frac{1}{(n+2)(n+1)\cdots m_i}.$$

After these simplifications:

$$\prod_{1\leqslant i\leqslant n} \frac{n!}{m_i!} = \frac{\prod_{m_i<n} \quad (n-0)\,(n-1)\cdots(m_i+1)}{\prod_{m_i>n}(n+1)(n+2)\cdots m_i}.$$

Since $m_1 + \cdots + m_n = n^2$, the number of factors in the numerator is the same as that in the denominator, and since *each* factor upstairs is $\leqslant n$, while *each* factor downstairs is $\geqslant n+1$, the result is indeed < 1. □

A further application of Stirling's formula shows that, asymptotically as $n \longrightarrow \infty$:

$$\frac{(n^2)!}{n!\cdots n!} \sim n^{n^2-\frac{n}{2}+1} \frac{1}{(2\,\pi)^{\frac{n-1}{2}}} \frac{1}{e^{\frac{1}{12}}} \left[1 + \frac{31}{360\,n^2} + \frac{5287}{181\,440\,n^4} + \mathrm{O}\Big(\frac{1}{n^6}\Big)\right].$$

Terminology 2.2. Call the coefficient of $t_1^n \cdots t_n^n$ in $f_0(t)$:

$$\begin{aligned}\widetilde{I}_0 &:= \big[t_1^n \cdots t_n^n\big]\big(f_0(t)\big)\\ &= \frac{(n^2)!}{n!\cdots n!}\, a_1^n \cdots a_n^n\end{aligned}$$

the *central monomial.*

Since $a_1, \ldots, a_n \in \mathbb{N}^*$, this is a large integer. The notation $\widetilde{I}_0$ is borrowed from [6].

In fact, Appendices 1 and 2 of [6] provided almost all the details to verify that the choice of weights:

$$a_i := r^{n-i} \qquad (1 \leqslant i \leqslant n),$$

for some constant r independent of n, shall offer a degree bound in the Green-Griffiths conjecture of the form:

$$d \geqslant \mathsf{constant}^{\,n}.$$

which would improve the current $d \gtrsim n^n$ obtained in [6, 13].

For a certain nefness condition required to apply the holomorphic Morse inequalities, it is necessary to have at least:

$$r \geqslant 3.$$

It is also allowed to take r larger, for instance:

$$r = 9 \qquad \text{or} \qquad r = 12 \qquad \text{or} \qquad r = 20,$$

but one should try *not* to choose r increasing with n, like for instance $r = \sqrt{n}$, since the final degree bound would otherwise be (explanations will appear later):

$$d \gtrsim (\sqrt{n})^n \gg \mathsf{constant}^n.$$

In [6], the choice was $r := n$, and this conducted to $d \gtrsim n^n$.

With a fixed (bounded) constant $r \geqslant 3$, the final degree bound for Green-Griffiths will be close to:

$$d \gtrsim \big(r\,(1+\varepsilon(r))\big)^n = \mathsf{constant}^n,$$

as we will verify in details later. The only remaining substantial piece of work to be done is to solve the following

Problem 2.3. *With the choice of weights:*

$$a_1 := r^{n-1}, \quad a_2 := r^{n-2}, \quad \ldots\ldots, \quad a_{n-1} := r, \quad a_n := 1,$$

to show that the coefficient of the monomial $t_1^n \cdots t_n^n$ in the product $C(t) \cdot f_0(t)$, namely:

$$I_0 := \big[t_1^n \cdots t_n^n\big]\Big(C(t_1,\ldots,t_n)\cdot f_0(t_1,\ldots,t_n)\Big)$$

is at least equal to the central monomial:

$$\begin{aligned} I_0 &\overset{?}{\geqslant} \widetilde{I}_0 \\ &= \frac{(n^2)!}{(n!)^n}\, r^{n\frac{n(n-1)}{2}}. \end{aligned}$$

In fact, several computer experiments convince that instead of $\frac{I_0}{\widetilde{I}_0} \geqslant 1$, a better inequality seems to hold:

$$\frac{I_0}{\widetilde{I}_0} \gtrsim \big(\mathsf{constant}_r\big)^n,$$

for some $\mathsf{constant}_r > 1$ which depends on r, and is closer and closer to 1 when r increases. So experimentally, $I_0 \geqslant \widetilde{I}_0$ is more than true. The goal is to set up a proof.

We start in Section **3** by verifying that a proof of $\frac{I_0}{\widetilde{I}_0} \geqslant 1$ implies a degree bound for Green-Griffiths of the announced form $d \geqslant \mathsf{constant}^n$; this task was already almost completely performed by Darondeau in [6].

Then in subsequent sections, we study the product $C(t_1,\ldots,t_n)$ and we establish $I_0 \geqslant \widetilde{I}_0$.

3. End of Proof of Theorem 1.3

It essentially suffices to read Appendices 1 and 2 of [6], with in mind that Darondeau's (simplifying) choice:

$$a_i := n^{n-i} \qquad (1 \leqslant i \leqslant n),$$

should be replaced with the choice:

$$a_i := r^{n-i} \qquad (1 \leqslant i \leqslant n),$$

where $r \geqslant 3$ is a fixed constant. Later, we will see that the choice $r = 3$ might expose to some computational difficulties, while as soon that $r \geqslant 9$, a serendipitous positivity property occurs. In any case, the estimates of the mentioned Appendix 2 were prepared in advance to work for any choice of $r = 3, 9, 12, 20, \log n, \sqrt{n}, n$, while they were applied

in [6] to $r = n$ by lack of a solution to Problem 2.3. Before solving this problem in the next sections, let us admit temporarily that it has a positive answer for a certain fixed:

$$9 \leqslant r \leqslant 20 \qquad \text{(hypothesis throughout)}.$$

End of proof of Theorem 1.3. In the notations of [6], the lower degree bound:

$$d \geqslant \boldsymbol{d}_{\mathrm{GG}}(n)$$

is determined by the largest root of a certain polynomial equation:

$$d^n\, I_0 + d^{n-1}\, I_1 + \cdots + d^{n-p}\, I_p + \cdots + I_n \;=\; 0,$$

with $I_0 > 0$. Of course, I_0 is the same as in Problem 2.3, hence we assume temporarily not only that it is positive, but also that it is quite large:

$$\frac{I_0}{\widetilde{I}_0} \geqslant 1.$$

We refer to [6] for a presentation of the other coefficients I_p.

Proposition 3.1. *The polynomial in the degree d of a hypersurface $\mathbb{X}^n \subset \mathbb{P}^{n-1}(\mathbb{C})$:*

$$d^n\, I_0 + d^{n-1}\, I_1 + \cdots + d^{n-p}\, I_p + \cdots + d\, I_{n-1} + I_n$$

takes positive values for all degrees:

$$\begin{aligned} d &\geqslant 25\, n^2 \cdot \big(r+3\big)^n. \\ &=: \boldsymbol{d}_{\mathrm{GG}}(n,r). \end{aligned}$$

In fact, a glance at the end of the proof below shows a slightly better, though more complicated:

$$\boldsymbol{d}_{\mathrm{GG}}(n,r) \;:=\; \big(20\, n^2 + 4\, n\big) \cdot \frac{r^3}{(r-1)^3\,(r+3)} \cdot \big(r+3\big)^n.$$

Theorem 1.3 terminates by checking on a computer that:

$$2^{\,5n} \;\geqslant\; \big(20\, n^2 + 4\, n\big) \cdot \frac{r^3}{(r-1)^3\,(r+3)} \cdot \big(r+3\big)^n \qquad (\forall\, n \geqslant 20),$$

for any choice of $9 \leqslant r \leqslant 20$. □

Proof of Proposition 3.1. In [6], the pole order of so-called *slanted vector fields* $c_n := n(n+2)$ is used. But the article [5] improves it to:

$$c_n \;:=\; 5\, n - 2.$$

Then with $c := c_n + 1$, the quantity $\frac{c+2}{2}$ appears several times in [6], so we may read:

$$\frac{c+2}{2} \;=\; \frac{5\, n + 1}{2}.$$

Next, with:

$$a_i \;:=\; r^{n-i} \qquad (1 \leqslant i \leqslant n),$$

set:

$$\mu(a) \;:=\; 1\, a_1 + 2\, a_2 + \cdots + n\, a_n,$$

and for all $1 \leqslant p \leqslant n$, set:

$$\widetilde{I}_p \;:=\; \underbrace{\frac{(n^2)!}{(n!)^n}\, a_1^n \cdots a_n^n}_{\text{recognize } \widetilde{I}_0} \, \big(2\,n\,\mu(a)\big)^p \sum_{1\leqslant i_1<\cdots<i_p\leqslant n} \frac{1}{a_{i_1}} \,\cdots\, \frac{1}{a_{i_p}}.$$

Importantly, Lemma A.6 on page 1919 of Appendix 2 shows that:

$$\frac{|I_p|}{\widetilde{I}_p} \;\leqslant\; \frac{5n+1}{2} \cdot |B|\Big(\frac{2n\mu(a)h}{a_1}, \ldots, \frac{2n\mu(a)h}{a_n}\Big) \cdot |C|\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\Big) \qquad (1\leqslant p\leqslant n).$$

It is not necessary to dwell into details about the middle quantity $|B|$, since Lemma A.7 on page 1920 shows that for any choice of weights $a_1, \ldots, a_n$:

$$\begin{aligned} |B|\Big(\frac{2n\mu(a)h}{a_1}, \ldots, \frac{2n\mu(a)h}{a_n}\Big) \;&\leqslant\; \Big(\frac{2\,n}{2\,n-1}\Big)^{n+1} \\ &\leqslant\; 2 \qquad (\forall\, n\geqslant 4 \text{ – exercise}). \end{aligned}$$

Consequently, we get:

$$\frac{|I_p|}{\widetilde{I}_p} \;\leqslant\; \big(5\,n+1\big) \cdot |C|\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\Big) \qquad (1\leqslant p\leqslant n).$$

Next, page 1914 uses the control:

$$|C|\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\Big) \;\leqslant\; \widehat{C}\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\Big),$$

by the 'majorant' series:

$$\widehat{C}\big(t_1, \ldots, t_n\big) \;:=\; \prod_{1\leqslant i<j\leqslant n} \frac{t_j - t_i}{t_j - 2\,t_i} \prod_{2\leqslant i<j\leqslant n} \frac{t_j - 2\,t_i}{t_j - 2\,t_i - t_{i-1}}.$$

Replacing the formal variables by the inverses of the weights, we get:

$$\widehat{C}\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_{n-1}}, \frac{1}{a_n}\Big) \;=\; \prod_{1\leqslant i<j\leqslant n} \frac{a_i/a_j - 1}{a_i/a_j - 2} \prod_{2\leqslant i<j\leqslant n} \frac{a_i/a_j - 2}{a_i/a_j - 2 - a_i/a_{i-1}}.$$

Since $a_i = r^{n-i}$ for $1 \leqslant i \leqslant n$, this rewrites as:

$$\begin{aligned} \widehat{C}\Big(\frac{1}{r^{n-1}}, \ldots, \frac{1}{r}, \frac{1}{1}\Big) \;&=\; \prod_{1\leqslant i<j\leqslant n} \frac{r^{j-i} - 1}{r^{j-i} - 2} \prod_{2\leqslant i<j\leqslant n} \frac{r^{j-i} - 2}{r^{j-i} - 2 - \frac{1}{r}} \\ \text{[Extract } i=1\text{]} \qquad &=\; \prod_{2\leqslant j\leqslant n} \frac{r^{j-1} - 1}{r^{j-1} - 2} \prod_{2\leqslant i<j\leqslant n} \Big[\, \frac{r^{j-i} - 1}{\underline{r^{j-i} - 2}_{\circ}} \, \frac{\underline{r^{j-i} - 2}_{\circ}}{r^{j-i} - 2 - \frac{1}{r}} \,\Big] \\ \text{[Simplify]} \qquad &=\; \prod_{2\leqslant j\leqslant n} \frac{r^{j-1} - 1}{r^{j-1} - 2} \prod_{2\leqslant i<j\leqslant n} \frac{r^{j-i+1} - r}{r^{j-i+1} - 2\,r - 1} \,\Big] \\ \text{[Rename indices]} \qquad &=\; \prod_{1\leqslant k\leqslant n-1} \frac{r^k - 1}{r^k - 2} \prod_{2\leqslant \ell\leqslant n-1} \Big(\frac{r^\ell - r}{r^\ell - 2\,r - 1}\Big)^{n-\ell}. \end{aligned}$$

Using inequalities valid as soon as $r \geqslant 4$ hence for $r \geqslant 9$:

$$\frac{1}{r^k - 2} \;\leqslant\; \frac{2}{r^k} \qquad (\forall\, k\geqslant 1),$$

and using the classical inequality $\log(1+\varepsilon) \leqslant \varepsilon$ valid for $-1 < \varepsilon < \infty$, let us show that the first product is bounded by a universal constant, and even by a constant which decreases as r increases:

$$\begin{aligned}
\prod_{1\leqslant k\leqslant n-1} \Big(1+\frac{1}{r^k-2}\Big) &\leqslant \prod_{k=1}^{\infty} \Big(1+\frac{2}{r^k}\Big) \\
&= \exp\bigg(\sum_{k=1}^{\infty} \log\Big(1+\frac{2}{r^k}\Big)\bigg) \\
[\log(1+\varepsilon)\leqslant\varepsilon] \qquad &\leqslant \exp\bigg(\frac{2}{r}\sum_{k=0}^{\infty}\frac{1}{r^k}\bigg) \\
&= \exp\Big(\frac{2}{r-1}\Big) \\
&\leqslant 1+\frac{3}{r},
\end{aligned}$$

because:

$$\begin{aligned}
\exp\Big(\frac{2}{r-1}\Big) &= 1+\frac{2}{r-1}+\frac{1}{2!}\frac{2^2}{(r-1)^2}+\sum_{k\geqslant 3}\frac{1}{k!}\frac{2^k}{(r-1)^k} \\
[\tfrac{2^k}{k!}\leqslant 2 \text{ for } k\geqslant 3] \qquad &\leqslant 1+\frac{2}{r-1}+\frac{2}{(r-1)^2}+2\frac{1}{(r-1)^3}\frac{1}{1-\frac{1}{r-1}} \\
&= 1+\frac{2}{r-1}+\frac{2}{(r-1)^2}\Big(1+\frac{1}{r-2}\Big) \\
&= 1+\frac{2}{r-1}\Big(1+\frac{1}{r-2}\Big) \\
&= 1+\frac{2}{r-2} \\
[\forall\, r\geqslant 6] \qquad &\leqslant 1+\frac{3}{r}.
\end{aligned}$$

The second product is bounded by a constant power $n-2$:

$$\begin{aligned}
\prod_{2\leqslant\ell\leqslant n-1}\Big(\frac{r^\ell-r}{r^\ell-2\,r-1}\Big)^{n-\ell} &= \prod_{2\leqslant\ell\leqslant n-1}\Big(1+\frac{r+1}{r^\ell-2\,r-1}\Big)^{n-\ell} \\
&\leqslant \bigg(\prod_{2\leqslant\ell\leqslant n-1}\Big(1+\frac{r+1}{r^\ell-2\,r-1}\Big)\bigg)^{n-2} \\
&\leqslant \bigg(\prod_{\ell=2}^{\infty}\Big(1+\frac{r+1}{r^\ell-2\,r-1}\Big)\bigg)^{n-2}.
\end{aligned}$$

Let us estimate this constant, which depends on r:

$$\alpha(r) := \prod_{\ell=2}^{\infty}\Big(1+\frac{r+1}{r^\ell-2\,r-1}\Big).$$

Lemma 3.2. *For all $r \geqslant 6$, one has:*

$$1 + \frac{r+1}{r^\ell - 2\,r - 1} \;\leqslant\; 1 + \frac{2}{r^{\ell-1}} \qquad (\forall\, \ell \geqslant 2).$$

Proof. This amounts to:

$$4\,r + 2 \overset{?}{\leqslant} r^2 - r \overset{\text{OK}}{\leqslant} r^\ell - r^{\ell-1} \qquad (\forall\, r \geqslant 6,\, \forall\, \ell \geqslant 2).$$

The first inequality $0 \overset{?}{\leqslant} r^2 - 5\,r - 2$ is true since $\frac{5+\sqrt{33}}{2} = 5.37\cdots < 6$. □

Hence we can majorize still assuming $r \geqslant 9$ throughout:

$$\begin{aligned}
\alpha(r) &\leqslant \prod_{\ell=2}^{\infty} \Big(1 + \frac{2}{r^{\ell-1}}\Big) \\
&= \exp\Big(\sum_{\ell=0}^{\infty} \log\Big(1 + \frac{2}{r^{\ell+1}}\Big)\Big) \\
&\leqslant \exp\Big(\frac{2}{r}\sum_{\ell=0}^{\infty} \frac{1}{r^\ell}\Big) \qquad [\log(1+\varepsilon) \leqslant 1] \\
&= \exp\Big(\frac{2}{r-1}\Big) \\
&\leqslant 1 + \frac{3}{r}. \qquad [\text{Seen above}]
\end{aligned}$$

In summary, we have shown that:

$$\frac{|I_p|}{\widetilde{I}_p} \;\leqslant\; \big(5\,n+1\big)\cdot\Big(1+\frac{3}{r}\Big)\cdot\Big(1+\frac{3}{r}\Big)^{n-2} \qquad (\forall\, n \geqslant 2).$$

Next, we estimate, still with $a_i = r^{n-i}$ for $i = 1, \ldots, n$:

$$\begin{aligned}
\mu(a) &= 1\cdot a_1 + 2\cdot a_2 + \cdots + (n-1)\,a_{n-1} + n\,a_n \\
&= 1\,r^{n-1} + 2\,r^{n-2} + \cdots + (n-1)\,r^1 + n\,r^0 \\
&= (n+1)\,\big[r^{n-1} + r^{n-2} + \cdots + r^1 + r^0\big] \\
&\quad - n\,r^{n-1} - (n-1)\,r^{n-2} - \cdots - 2\,r^1 - 1\,r^0 \\
&= (n+1)\,\frac{r^n - 1}{r-1} \\
&\quad - \frac{n\,r^{n+1} - (n+1)\,r^n + 1}{(r-1)^2},
\end{aligned}$$

the result in this last line being obtained simply by differentiating with respect to r the classical:

$$r^n + r^{n-1} + \cdots + r^2 + r + 1 \;=\; \frac{r^{n+1}-1}{r-1}.$$

A reduction to the same denominator contracts:

$$\begin{aligned}\mu(a) &= \frac{r^{n+1}-(n+1)\,r+n}{(r-1)^2}\\ &\leqslant \frac{r^{n+1}}{(r-1)^2}\\ &= \frac{r}{(r-1)^2}\,r^n.\end{aligned}$$

Next, consider generally a polynomial of degree $n \geqslant 1$ with complex coefficients $c_p \in \mathbb{C}$:

$$c_0\,z^n + c_1\,z^{n-1} + \cdots + c_{n-1}\,z^1 + c_n \qquad (c_0 \neq 0).$$

Theorem 3.3. [Fujiwara] *The moduli of all roots of $c_0 z^n + c_1 z^{n-1} + \cdots + c_n$ are bounded by:*

$$\max\big|\text{roots}\big| \;\leqslant\; 2\,\max_{1\leqslant p\leqslant n}\sqrt[p]{\frac{|c_p|}{|c_0|}}\,. \qquad \square$$

Now, come back to the polynomial $d^n\,I_0 + d^{n-1}\,I_1 + \cdots + I_n$ of Proposition 3.1. Thanks to Fujiwara:

$$\begin{aligned}\max\big|\text{roots}\big| &\leqslant 2\,\max_{1\leqslant p\leqslant n}\sqrt[p]{\frac{|I_p|}{I_0}}\\ &\qquad [I_0 \geqslant \widetilde{I}_0 \text{ is assumed}]\\ &\leqslant 2\,\max_{1\leqslant p\leqslant n}\sqrt[p]{\frac{|I_p|}{\widetilde{I}_0}}\\ &= 2\,\max_{1\leqslant p\leqslant n}\sqrt[p]{\frac{|I_p|}{\widetilde{I}_p}\cdot\frac{\widetilde{I}_p}{\widetilde{I}_0}}\\ &\qquad [\text{Seen above}]\\ &\leqslant 2\,\max_{1\leqslant p\leqslant n}\sqrt[p]{\big(5\,n+1\big)\Big(1+\frac{3}{r}\Big)^{n-1}\cdot\frac{\widetilde{I}_p}{\widetilde{I}_0}}\\ &\leqslant 2\,\max_{1\leqslant p\leqslant n}\sqrt[p]{\big(5\,n+1\big)\Big(1+\frac{3}{r}\Big)^{n-1}}\cdot\max_{1\leqslant p\leqslant n}\sqrt[p]{\frac{\widetilde{I}_p}{\widetilde{I}_0}}\\ &= 2\,\big(5\,n+1\big)\Big(1+\frac{3}{r}\Big)^{n-1}\cdot\max_{1\leqslant p\leqslant n}\sqrt[p]{\frac{\widetilde{I}_p}{\widetilde{I}_0}}\,\cdot\end{aligned}$$

Next, coming back to the definition of $\widetilde{I}_p$, it remains to estimate the p-th roots of the quotients:

$$\frac{\widetilde{I}_p}{\widetilde{I}_0} \;=\; \big(2n\,\mu(a)\big)^p\;\underbrace{\sum_{1\leqslant i_1<\cdots<i_p\leqslant n}\frac{1}{a_{i_1}}\,\cdots\,\frac{1}{a_{i_p}}}_{=:\;\sigma_p(\frac{1}{a_1},\ldots,\frac{1}{a_n})},$$

which incorporate the p-th symmetric functions σ_p of the weight inverses $\frac{1}{a_i}$. We start by extracting the p-th root of $\big(2n\,\mu(a)\big)^p$ easily:

$$\begin{aligned}\max_{1\leqslant p\leqslant n}\sqrt[p]{\frac{\widetilde{I}_p}{\widetilde{I}_0}} \;&=\; 2n\,\mu(a)\cdot\max_{1\leqslant p\leqslant n}\sqrt[p]{\sigma_p\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big)}\\ \text{[Seen above]}\qquad\qquad &\leqslant\; 2n\,\frac{r}{(r-1)^2}\,r^n\cdot\max_{1\leqslant p\leqslant n}\sqrt[p]{\sigma_p\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big)}.\end{aligned}$$

Lemma 3.4. *One has:*

$$\max_{1\leqslant p\leqslant n}\sqrt[p]{\sigma_p\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big)}\;=\;\sigma_1\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big).$$

Proof. For positive real numbers $b_1,\ldots,b_n>0$, the renormalized symmetric functions:

$$\begin{aligned}s_p\big(b_1,\ldots,b_n\big)\;&:=\;\frac{1}{\binom{n}{p}}\sum_{1\leqslant i_1<\cdots<i_p\leqslant n}b_{i_1}\cdots b_{i_p}\\ &=\;\frac{1}{\binom{n}{p}}\,\sigma_p\,\big(b_1,\ldots,b_p\big),\end{aligned}$$

satisfy the classical *Mac Laurin inequality:*

$$s_1\;\geqslant\;\sqrt[2]{s_2}\;\geqslant\;\sqrt[3]{s_3}\;\geqslant\;\cdots\cdots\;\geqslant\;\sqrt[n]{s_n}.$$

A modified version, useful to us, is:

Assertion 3.5. *A similar, less fine, inequality, holds before renormalization:*

$$\sigma_1\;\geqslant\;\sqrt[2]{\sigma_2}\;\geqslant\;\cdots\cdots\;\geqslant\;\sqrt[p]{\sigma_p}\;\geqslant\;\sqrt[p+1]{\sigma_{p+1}}\;\geqslant\;\cdots\cdots\;\geqslant\;\sqrt[n]{\sigma_n}.$$

Proof. For $1\leqslant p\leqslant n-1$, we would deduce from Mac Laurin what we want:

$$\left(\sqrt[p]{\frac{\sigma_p}{\binom{n}{p}}}\overset{\text{known}}{\geqslant}\sqrt[p+1]{\frac{\sigma_{p+1}}{\binom{n}{p+1}}}\right)\quad\Longrightarrow\quad\Big(\sqrt[p]{\sigma_p}\overset{?}{\geqslant}\sqrt[p+1]{\sigma_{p+1}}\Big),$$

provided it would be true that:

$$\frac{\sqrt[p]{\binom{n}{p}}}{\sqrt[p+1]{\binom{n}{p+1}}}\;\overset{?}{\geqslant}\;1\qquad\qquad(\forall\,1\leqslant p\leqslant n-1).$$

We claim that such numerical inequalities hold true. Indeed, from the two visible minorations:

$$\begin{aligned}n\,(n-1)\,\cdots\,(n-p+1)\;&\geqslant\;\big(n-p\big)^p,\\ \big(p+1\big)^p\;&\geqslant\;1\cdot 2\cdot\ldots\cdot p,\end{aligned}$$

it comes:

$$\frac{n\,(n-1)\,\cdots\,(n-p+1)}{1\cdot 2\cdot\ldots\cdot p}\;\geqslant\;\frac{(n-p)^p}{(p+1)^p},$$

whence we obtain what we wanted:

$$\left[\frac{n\,(n-1)\,\cdots\,(n-p+1)}{1\cdot 2\cdot\ldots\cdot p}\right]^{p+1}\;\geqslant\;\left[\frac{n\,(n-1)\,\cdots\,(n-p+1)}{1\cdot 2\cdot\ldots\cdot p}\cdot\frac{(n-p)}{(p+1)}\right]^{p}.\qquad\square$$

Lastly, with $b_1 := \frac{1}{a_1}, \ldots, b_n := \frac{1}{a_n}$, we get:

$$\sigma_1\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\Big) \ \geqslant \ \sqrt[p]{\sigma_p\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\Big)} \qquad (\forall\, 1 \leqslant p \leqslant n),$$

which forces the maximum to be attained precisely when $p = 1$.

So we obtain:

$$\max_{1 \leqslant p \leqslant n} \sqrt[p]{\frac{\widetilde{I}_p}{\widetilde{I}_0}} \ \leqslant \ 2n \, \frac{r}{(r-1)^2} \, r^n \cdot \sigma_1\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\Big).$$

and it only remains to estimate:

$$\begin{aligned} \sigma_1\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_{n-1}}, \, \frac{1}{a_n}\Big) \ &\leqslant \ \frac{1}{r^{n-1}} + \cdots + \frac{1}{r} + 1 \\ &\leqslant \ \frac{r}{r-1}, \end{aligned}$$

in order to finish the proof of Proposition 3.1:

$$\begin{aligned} \max \big|\mathsf{roots}\big| \ &\leqslant \ \big(10\,n + 2\big) \Big(1 + \frac{3}{r}\Big)^{n-1} \cdot \max_{1 \leqslant p \leqslant n} \sqrt[p]{\frac{\widetilde{I}_p}{\widetilde{I}_0}} \\ &\leqslant \ \big(10\,n + 2\big) \Big(1 + \frac{3}{r}\Big)^{n-1} \cdot 2n \, \frac{r}{(r-1)^2} \, r^n \cdot \sigma_1\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\Big) \\ &\leqslant \ \big(10\,n + 2\big) \, \frac{(r+3)^{n-1}}{r^{n-1}} \cdot 2n \, \frac{r}{(r-1)^2} \, r^n \cdot \frac{r}{r-1} \\ &= \ \big(20\,n^2 + 4\,n\big) \cdot \frac{r^3}{(r-1)^3\,(r+3)} \cdot \big(r+3\big)^n \\ &\leqslant \ 25\,n^2 \cdot \big(r+3\big)^n. \end{aligned}$$

□

4. From Coordinates $(t_1, t_2, \ldots, t_n)$ to Coordinates $(w_2, \ldots, w_n)$

The goal of this section is to transform both the product $C(t)$ and the n^2-power $f_0(t)$ into more tractable expressions, by introducing the formal variables:

$$w_2 \ := \ \frac{t_1}{t_2}, \quad w_3 \ := \ \frac{t_2}{t_3}, \quad \ldots\ldots, \quad w_n \ := \ \frac{t_{n-1}}{t_n}.$$

To enhance intuition, start by expanding the writing of the factors of two types in the considered double big product:

$$\begin{aligned} C(t_1,\ldots,t_n) \;=\; & \frac{t_2-t_1}{t_2-2\,t_1}\,\frac{t_3-t_1}{t_3-2\,t_1}\,\cdots\,\frac{t_n-t_1}{t_n-2\,t_1} \\ & \qquad\qquad \frac{t_3-t_2}{t_3-2\,t_2}\,\cdots\,\frac{t_n-t_2}{t_n-2\,t_2} \\ & \qquad\qquad\qquad\qquad \ddots \quad \vdots \\ & \qquad\qquad\qquad\qquad \frac{t_n-t_{n-1}}{t_n-2\,t_{n-1}} \\ & \qquad \frac{t_3-2\,t_2}{t_3-2\,t_2+t_1}\,\cdots\,\frac{t_n-2\,t_2}{t_n-2\,t_2+t_1} \\ & \qquad\qquad\qquad\qquad \ddots \quad \vdots \\ & \qquad\qquad\qquad\qquad \frac{t_n-2\,t_{n-1}}{t_n-2\,t_{n-1}+t_{n-2}}. \end{aligned}$$

To pass to the new variables, compute first for instance:

$$\begin{aligned} \frac{t_2-t_1}{t_2-2\,t_1} \;&=\; \frac{1-\frac{t_1}{t_2}}{1-2\,\frac{t_1}{t_2}} \;=\; \frac{1-w_2}{1-2\,w_2}, \\ \frac{t_3-t_1}{t_3-2\,t_1} \;&=\; \frac{1-\frac{t_1}{t_3}}{1-2\,\frac{t_1}{t_3}} \;=\; \frac{1-\frac{t_1}{t_2}\,\frac{t_2}{t_3}}{1-2\,\frac{t_1}{t_2}\,\frac{t_2}{t_3}} \;=\; \frac{1-w_2w_3}{1-2\,w_2w_3}, \\ \frac{t_5-2\,t_2}{t_5-2\,t_2+t_1} \;&=\; \frac{1-2\,\frac{t_2}{t_3}\,\frac{t_3}{t_4}\,\frac{t_4}{t_5}}{1-2\,\frac{t_2}{t_3}\,\frac{t_3}{t_4}\,\frac{t_4}{t_5}+\frac{t_1}{t_2}\,\frac{t_2}{t_3}\,\frac{t_3}{t_4}\,\frac{t_4}{t_5}} \;=\; \frac{1-2\,w_3w_4w_5}{1-2\,w_3w_4w_5+w_2w_3w_4w_5}. \end{aligned}$$

Generally, with as above:

$$w_i \;:=\; \frac{t_{i-1}}{t_i} \qquad (2\leqslant i\leqslant n),$$

we can transform all the factors of first type, for indices $2 \leqslant i \leqslant j \leqslant n$ — mind the shift $i \longmapsto i-1$ from the original definition of $C(t)$:

$$\begin{aligned} E_{i,j}(t) \;:=\; \frac{t_j-t_{i-1}}{t_j-2\,t_{i-1}} \;=\; \frac{1-\frac{t_{i-1}}{t_j}}{1-2\,\frac{t_{i-1}}{t_j}} \;&=\; \frac{1-\frac{t_{i-1}}{t_i}\cdots\frac{t_{j-1}}{t_j}}{1-2\,\frac{t_{i-1}}{t_i}\cdots\frac{t_{j-1}}{t_j}} \\ &=\; \frac{1-w_i\cdots w_j}{1-2\,w_i\cdots w_j} \\ &=:\; E_{i,j}(w). \end{aligned}$$

Similarly, for $3 \leqslant i \leqslant j \leqslant n$, again with the shift $i \longmapsto i-1$:

$$\begin{aligned} F_{i,j}(t) \;:=\; \frac{t_j-2\,t_{i-1}}{t_j-2\,t_{i-1}+t_{i-2}} \;=\; \frac{1-2\,\frac{t_{i-1}}{t_j}}{1-2\,\frac{t_{i-1}}{t_j}+\frac{t_{i-2}}{t_j}} \;&=\; \frac{1-2\,\frac{t_{i-1}}{t_i}\cdots\frac{t_{j-1}}{t_j}}{1-2\,\frac{t_{i-1}}{t_i}\cdots\frac{t_{j-1}}{t_j}+\frac{t_{i-2}}{t_{i-1}}\frac{t_{i-1}}{t_i}\cdots\frac{t_{j-1}}{t_j}} \\ &=\; \frac{1-2\,w_i\cdots w_j}{1-2\,w_i\cdots w_j+w_{i-1}w_i\cdots w_j} \\ &=:\; F_{i,j}(w). \end{aligned}$$

Consequently:

$$\begin{aligned} C(t_1,t_2,\ldots,t_n) \,=\, C(w_2,\ldots,w_n) \,:=\, & \frac{1-w_2}{1-2\,w_2}\,\frac{1-w_2w_3}{1-2\,w_2w_3}\,\cdots\,\frac{1-w_2w_3\cdots w_n}{1-2\,w_2w_3\cdots w_n} \\ & \qquad\qquad \frac{1-w_3}{1-2\,w_3}\,\cdots\,\frac{1-w_3\cdots w_n}{1-2\,w_3\cdots w_n} \\ & \qquad\qquad\qquad\qquad \ddots \qquad \vdots \\ & \qquad\qquad\qquad\qquad\qquad\qquad \frac{1-w_n}{1-2\,w_n} \\ & \frac{1-2\,w_3}{1-2\,w_3+w_2w_3}\,\cdots\,\frac{1-2\,w_3\cdots w_n}{1-2\,w_3\cdots w_n+w_2w_3\cdots w_n} \\ & \qquad\qquad\qquad\qquad \ddots \qquad \vdots \\ & \qquad\qquad\qquad\qquad\qquad\qquad \frac{1-2\,w_n}{1-2\,w_n+w_{n-1}w_n}. \end{aligned}$$

This can be abbreviated as:

$$\begin{aligned} C(w) \,&=\, \prod_{2\leqslant i\leqslant j\leqslant n} \frac{1-w_i\cdots w_j}{1-2\,w_i\cdots w_j}\,\prod_{3\leqslant i\leqslant j\leqslant n} \frac{1-2\,w_i\cdots w_j}{1-2\,w_i\cdots w_j+w_{i-1}w_i\cdots w_j} \\ &=\, \prod_{2\leqslant i\leqslant j\leqslant n} E'_{i,j}\,\prod_{3\leqslant i\leqslant j\leqslant n} F'_{i,j}. \end{aligned}$$

As is visible — and as was already visible before in variables $(t_1,\ldots,t_n)$ —, the terms $1-2\,w_i\cdots w_j$ that appear in the denominators of the $E'_{i,j}$ cancel out with the same terms appearing in the numerators of the $F'_{i,j}$, though only for $3\leqslant i\leqslant j\leqslant n$. These simplifications conduct to the shorter representation:

$$\begin{aligned} C(w_2,\ldots,w_n) \,:=\, & \frac{1-w_2}{1-2\,w_2} \quad \frac{1-w_2w_3}{1-2\,w_2w_3} \quad \cdots\cdots\cdots\cdots\cdots \quad \frac{1-w_2w_3\cdots w_n}{1-2\,w_2w_3\cdots w_n} \\ & \qquad\qquad \frac{1-w_3}{1-2\,w_3+w_2w_3} \quad \cdots\cdots \quad \frac{1-w_3\cdots w_n}{1-2\,w_3\cdots w_n+w_2w_3\cdots w_n} \\ & \qquad\qquad\qquad\qquad \ddots \qquad\qquad\qquad \vdots \\ & \qquad\qquad\qquad\qquad\qquad\quad \ddots \qquad\qquad \vdots \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1-w_n}{1-2\,w_n+w_{n-1}w_n}, \end{aligned}$$

which can be re-abbreviated as:

$$\begin{aligned} C(w_2,\ldots,w_n) \,=\, & E_2(w_2)\,E_3(w_2,w_3)\,\cdots E_n(w_2,w_3,\ldots,w_n) \\ & \qquad F_{3,3}(w_2,w_3)\cdots F_{3,n}(w_2,w_3,\ldots,w_n) \\ & \qquad\qquad \ddots \qquad\qquad \vdots \\ & \qquad\qquad\qquad\qquad F_{n,n}(w_{n-1},w_n), \end{aligned}$$

that is to say:

$$C(w_2,\ldots,w_n) \;=\; \prod_{2\leqslant j\leqslant n} \underbrace{\frac{1-w_2\cdots w_j}{1-2\,w_2\cdots w_j}}_{=:\,E_j} \prod_{3\leqslant i\leqslant j\leqslant n} \underbrace{\frac{1-w_i\cdots w_j}{1-2\,w_i\cdots w_j+w_{i-1}w_i\cdots w_j}}_{=:\,F_{i,j}}.$$

Next, let us re-express in the w_i variables:

$$\begin{aligned} f_0(t) \;&=\; \Big(a_1t_1+\cdots+a_{n-2}t_{n-2}+a_{n-1}t_{n-1}+a_nt_n\Big)^{n^2} \\ &=\; \Big(a_1\tfrac{t_1}{t_n}+\cdots+a_{n-2}\tfrac{t_{n-2}}{t_n}+a_{n-1}\tfrac{t_{n-1}}{t_n}+a_n\tfrac{t_n}{t_n}\Big)^{n^2}\,\big(t_n\big)^{n^2} \\ &=\; \Big(r^{n-1}w_2\cdots w_n+\cdots+r^2w_{n-1}w_n+r\,w_n+1\Big)^{n^2}\,\big(t_n\big)^{n^2}. \end{aligned}$$

To yet transform $t_n^{n^2}$ at the end, observe that:

$$\begin{aligned} \frac{1}{(w_2)^n(w_3)^{2n}\cdots(w_{n-1})^{n^2-2n}(w_n)^{n^2-n}} \;&=\; \frac{1}{\big(\frac{t_1}{t_2}\big)^n\,\big(\frac{t_2}{t_3}\big)^{2n}\cdots\big(\frac{t_{n-2}}{t_{n-1}}\big)^{n^2-2n}\big(\frac{t_{n-1}}{t_n}\big)^{n^2-n}} \\ &=\; \frac{1}{t_1^n\,t_2^n\,\cdots t_{n-2}^n\,t_{n-1}^n\,\frac{1}{t_n^{n^2-n}}} \\ &=\; \frac{t_n^{n^2}}{t_1^n\,t_2^n\,\cdots t_{n-2}^n\,t_{n-1}^n\,t_n^n}, \end{aligned}$$

whence:

$$f_0(t) \;=\; \frac{\big(r^{n-1}w_2\cdots w_n+\cdots+r^2w_{n-1}w_n+r\,w_n+1\big)^{n^2}}{w_2^n\,w_3^{2n}\,\cdots\,w_{n-1}^{n^2-2n}\,w_n^{n^2-n}}\;\frac{1}{t_1^n\,t_2^n\,\cdots\,t_{n-1}^n\,t_n^n}.$$

Consequently, in Problem 2.3, the coefficient I_0 of the monomial $t_1^n\cdots t_n^n$ in the product $C(t)\cdot f_0(t)$ identifies with the *constant* term, namely the coefficient of $w_2^0\cdots w_n^0=1$, in the product:

$$\frac{\big(r^{n-1}w_2\cdots w_n+\cdots+r^2w_{n-1}w_n+r\,w_n+1\big)^{n^2}}{w_2^n\,w_3^{2n}\,\cdots\,w_{n-1}^{n^2-2n}\,w_n^{n^2-n}}\cdot\begin{array}{ccccc} \frac{1-w_2}{1-2\,w_2} & \frac{1-w_2w_3}{1-2\,w_2w_3} & \cdots\cdots\cdots\cdots & \frac{1-w_2w_3\cdots w_n}{1-2\,w_2w_3\cdots w_n} \\ & \frac{1-w_3}{1-2\,w_3+w_2w_3} & \cdots\cdots & \frac{1-w_3\cdots w_n}{1-2\,w_3\cdots w_n+w_2w_3\cdots w_n} \\ & & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & \frac{1-w_n}{1-2\,w_n+w_{n-1}w_n} \end{array}.$$

It is now appropriate to expand the n^2 power in the numerator above plainly as:

$$\frac{1}{(w_2)^n\cdots(w_n)^{n^2-2n}}\sum_{\substack{i_1,\ldots,i_n\geqslant 0\\ i_1+\cdots+i_n=n^2}}(1)^{i_1}\,(rw_n)^{i_2}\big(r^2w_{n-1}w_n\big)^{i_3}\cdots\big(r^{n-1}w_2\cdots w_n\big)^{i_n}\,\frac{(n^2)!}{i_1!\,i_2!\,i_3!\,\cdots i_n!}.$$

Next, we would like to point out that $C(w_2,\ldots,w_n)$ is a product of rational expressions which expand all in converging power series at the origin. More precisely, using the trivial

expansion:

$$E(x) := \frac{1-x}{1-2\,x} = 1+\frac{x}{1-2\,x} = 1+\sum_{i=1}^{\infty} 2^{i-1}\,x^i,$$

together with the expansion of Lemma 6.1 — with the convention that $\binom{\ell-1}{-1} = 0 = \binom{\ell-1}{\ell}$ — :

$$F(x,y) := \frac{1-y}{1-2\,y+x\,y} = 1+\frac{y-x\,y}{1-2\,y+x\,y} = 1+\sum_{\ell=1}^{\infty} y^\ell \sum_{0\leqslant k\leqslant \ell} (-1)^k\,x^k\,\Big[2^{\ell-1-k}\,\tbinom{\ell-1}{k}+2^{\ell-k}\,\tbinom{\ell-1}{k-1}\Big],$$

and re-expressing:

$$C(w_2,\dots,w_n) = \prod_{2\leqslant j\leqslant n} E\big(w_2\cdots w_j\big) \prod_{3\leqslant i\leqslant j\leqslant n} F\big(w_{i-1},\,w_i\cdots w_j\big), \tag{4.1}$$

and lastly, multiplying all the obtained converging power series, one can in principle receive an expansion:

$$C(w_2,\dots,w_n) = \sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} C_{k_2,\dots,k_n}\,(w_2)^{k_2}\cdots(w_n)^{k_n},$$

which is holomorphic in a neighborhood of the origin. However, it is very delicate to reach closed explicit expressions for these integer Taylor coefficients $C_{k_2,\dots,k_n}$, a difficulty which lies at the very core of Problem 2.3.

In summary, the quantity I_0 we want to determine, in order to show that it satisfies $I_0 \geqslant \widetilde{I}_0$, is the coefficient of the constant term $w_2^0\cdots w_n^0$ in a product consisting of 2 rows:

$$\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} C_{k_2,\dots,k_n}\,(w_2)^{k_2}\cdots(w_n)^{k_n}\,\cdot$$

$$\cdot\,\frac{1}{(w_2)^n\cdots(w_n)^{n^2-n}} \sum_{\substack{i_1,\dots,i_n\geqslant 0\\ i_1+\cdots+i_n=n^2}} (1)^{i_1}\,(rw_n)^{i_2}\big(r^2w_{n-1}w_n\big)^{i_3}\cdots\big(r^{n-1}w_2\cdots w_n\big)^{i_n}\,\frac{(n^2)!}{i_1!\,i_2!\,i_3!\,\cdots i_n!}.$$

Clearly, the second row becomes, after reorganization, a Laurent series of the form:

$$\sum_{-n\leqslant\ell_2}\cdots\sum_{-(n^2-n)\leqslant\ell_n} J_{\ell_2,\dots,\ell_n}\,(w_2)^{\ell_2}\cdots(w_n)^{\ell_n}.$$

But because in the first row one always has $k_2,\dots,k_n \geqslant 0$, all Laurent monomials $(w_2)^{\ell_2}\cdots(w_n)^{\ell_n}$ in the second row for which $\ell_i \geqslant 1$ for some $1\leqslant i\leqslant n$ do *not* contribute to the determination of the desired constant term $w_2^0\cdots w_n^0$. So the summation in the second row can be truncated to:

$$\sum_{-n\leqslant\ell_2\leqslant 0}\cdots\sum_{-(n^2-n)\leqslant\ell_n\leqslant 0} J_{\ell_2,\dots,\ell_n}\,(w_2)^{\ell_2}\cdots(w_n)^{\ell_n}.$$

A supplementary change of indices followed by a reorganization conducts to an appropriate reformulation of what is I_0: the following statement will then constitute the very starting point of our further explorations.

Proposition 4.2. *One has:*

$$I_0 \,=\, \big[w_2^0 \cdots w_n^0\big]\,\Big(A(w_2,\dots,w_n)\cdot C(w_2,\dots,w_n)\Big),$$

where:

$$A(w_2,\dots,w_n) \,:=\, \sum_{\substack{0\leqslant k_2\leqslant n\\ 0\leqslant k_3\leqslant n+k_2\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ 0\leqslant k_{n-1}\leqslant n+k_{n-2}\\ 0\leqslant k_n\ \ \leqslant n+k_{n-1}}} \frac{(n^2)!}{(n-k_2)!(n+k_2-k_3)!\cdots(n+k_{n-2}-k_{n-1})!(n+k_{n-1}-k_n)!(n+k_n)!}\cdot$$

$$\cdot\,\frac{r^{\,n\frac{n(n-1)}{2}}}{r^{k_2+\cdots+k_n}}\,\frac{1}{(w_2)^{k_2}\cdots(w_n)^{k_n}},$$

and where $C(w_2,\dots,w_n)$ is as before.

Proof. We therefore rewrite:

$$\frac{1}{w_2^n w_3^{2n}\cdots w_{n-1}^{(n-2)n} w_n^{(n-1)n}} \sum_{\substack{i_1,\dots,i_n\geqslant 0\\ i_1+\cdots+i_n=n^2}} (1)^{i_1}\,(r\,w_n)^{i_2}\big(r^2 w_{n-1}w_n\big)^{i_3}\cdots\big(r^{n-2}w_3\cdots w_{n-1}w_n\big)^{i_{n-1}}\big(r^{n-1}w_2w_3\cdots w_{n-1}w_n\big)^{i_n}\cdot$$

$$\cdot\,\frac{(n^2)!}{i_1!\,i_2!\,i_3!\cdots i_{n-1}!i_n!} \,=$$

$$=:\, \sum_{-n\leqslant\ell_2}\ \sum_{-2n\leqslant\ell_3}\ \cdots\ \sum_{-(n-2)n\leqslant\ell_{n-1}}\ \sum_{-(n-1)n\leqslant\ell_n} J_{\ell_2,\ell_3,\dots,\ell_{n-1},\ell_n}\,(w_2)^{\ell_2}(w_3)^{\ell_3}\cdots(w_{n-1})^{\ell_{n-1}}(w_n)^{\ell_n},$$

so that the correspondence between exponents is:

$$\begin{aligned}
-\,n \,\leqslant\, \ell_2 \,&=\, i_n - n,\\
-\,2\,n \,\leqslant\, \ell_3 \,&=\, i_n + i_{n-1} - 2\,n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
-\,(n-2)\,n \,\leqslant\, \ell_{n-1} \,&=\, i_n + i_{n-1} + \cdots + i_3 - (n-2)\,n,\\
-\,(n-1)\,n \,\leqslant\, \ell_n \,&=\, i_n + i_{n-1} + \cdots + i_3 + i_2 - (n-1)\,n.
\end{aligned}$$

Performing the harmless truncations $\ell_2 \leqslant 0, \dots, \ell_n \leqslant 0$ leads then to the inequalities:

$$\begin{aligned}
0 \,\leqslant\, i_n \,&\leqslant\, n,\\
0 \,\leqslant\, i_n + i_{n-1} \,&\leqslant\, 2\,n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
0 \,\leqslant\, i_n + i_{n-1} + \cdots + i_3 \,&\leqslant\, (n-2)\,n,\\
0 \,\leqslant\, i_n + i_{n-1} + \cdots + i_3 + i_2 \,&\leqslant\, (n-1)\,n,
\end{aligned}$$

so that it suffices to consider, before multiplying by $C(w_2,\dots,w_n)$, the truncated series:

$$A := \sum_{\substack{i_1+\cdots+i_n=n^2\\ i_1\geqslant 0,\dots,i_n\geqslant 0\\ 0\leqslant i_n\leqslant n\\ 0\leqslant i_n+i_{n-1}\leqslant 2\,n\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ 0\leqslant i_n+i_{n-1}+\cdots+i_3\leqslant(n-2)\,n\\ 0\leqslant i_n+i_{n-1}+\cdots+i_3+i_2\leqslant(n-1)\,n}} \frac{1}{w_2^{n-i_n}\,w_3^{2n-i_{n-1}-i_n}\cdots w_{n-1}^{(n-2)n-i_n-i_{n-1}-\cdots-i_3}\,w_n^{(n-1)n-i_n-i_{n-1}-\cdots-i_3-i_2}}\cdot$$

$$\cdot\ r^{i_2+2\,i_3+\cdots+(n-2)i_{n-1}+(n-1)i_n}\ \cdot\ \frac{(n^2)!}{i_1!\,i_2!\,i_3!\,\cdots\,i_{n-1}!\,i_n!}.$$

To reach the expression shown by the proposition, introduce the new nonnegative integer indices:

$$\begin{aligned}
k_2 &:= n-i_n && (k_2\geqslant 0),\\
k_3 &:= 2\,n-i_{n-1}-i_n && (k_3\geqslant 0),\\
&\cdots\cdots\cdots\cdots\cdots\cdots && (\cdots\cdots\cdots),\\
k_{n-1} &:= (n-2)\,n-i_3-\cdots-i_{n-1}-i_n && (k_{n-1}\geqslant 0),\\
k_n &:= (n-1)\,n-i_2-i_3-\cdots-i_{n-1}-i_n && (k_n\geqslant 0).
\end{aligned}$$

To finish, three explanations are needed.

Firstly, one has the inequalities:

$$\begin{aligned}
0\ &\leqslant\ k_2\ \leqslant\ n,\\
0\ &\leqslant\ k_3\ \leqslant\ n+k_2,
\end{aligned}$$

because $i_n\geqslant 0$ and because:

$$k_3\ =\ n-i_{n-1}+n-i_n\ =\ \underbrace{n-i_{n-1}}_{i_{n-1}\geqslant 0}+k_2\ \leqslant\ n+k_2.$$

Similarly:

$$0\ \leqslant\ k_4\ =\ 3\,n-i_{n-2}-i_{n-1}-i_n\ =\ n-i_{n-2}+k_3\ \leqslant\ n+k_3,$$

and so on up to:

$$\begin{aligned}
0\ &\leqslant\ k_{n-1}\ \leqslant\ n+k_{n-2},\\
0\ &\leqslant\ k_n\ \leqslant\ n+k_{n-1}.
\end{aligned}$$

Secondly, since:

$$k_2+k_3+\cdots+k_{n-1}+k_n\ =\ n\,\big(1+2+\cdots+(n-2)+(n-1)\big)-i_2-2\,i_3-\cdots-(n-2)\,i_{n-1}-(n-1)\,i_n,$$

the exponent of r becomes:

$$i_2+2\,i_3+\cdots+(n-2)\,i_{n-1}+(n-1)\,i_n\ =\ n\,\tfrac{n(n-1)}{2}-k_2-k_3-\cdots-k_{n-1}-k_n.$$

Thirdly and lastly, the factorials become:

$$\begin{aligned}
i_n!\ &=\ (n-k_2)!,\\
i_{n-1}!\ &=\ \big(n+n-i_n-k_3\big)!\ =\ \big(n+k_2-k_3\big)!,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
i_2!\ &=\ \big(-\,k_n+(n-1)\,n-i_3-\cdots-i_n\big)!\ =\ \big(n+k_{n-1}-k_n\big)!,\\
i_1!\ &=\ \big(n\,n-(n+k_{n-1}-k_n)-\cdots-(n+k_2-k_3)-(n-k_2)\big)\ =\ \big(n+k_n\big)!.
\end{aligned}$$

These three explanations yield the expression of $A(w_2, \ldots, w_n)$ stated by the proposition. □

Next, because only the quotient $\frac{I_0}{\widetilde{I}_0}$ must be studied in order to reach the minoration $I_0 \geqslant \widetilde{I}_0$, we can divide everything in advance by the central monomial:

$$\widetilde{I}_0 \;=\; \frac{(n^2)!}{n! \cdots n!}\, r^{n\frac{n(n-1)}{2}}.$$

Equivalently, we factor:

$$A \;=\; \frac{(n^2)!}{n! \cdots n!}\, r^{n\frac{n(n-1)}{2}} \cdot$$
$$\cdot \sum_{\substack{0\leqslant k_2\leqslant n\\ 0\leqslant k_3\leqslant n+k_2\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ 0\leqslant k_{n-1}\leqslant n+k_{n-2}\\ 0\leqslant k_n \;\leqslant n+k_{n-1}}} \frac{n!}{(n-k_2)!}\, \frac{n!}{(n+k_2-k_3)!} \cdots \frac{n!}{(n+k_{n-2}-k_{n-1})!}\, \frac{n!}{(n+k_{n-1}-k_n)!(n+k_n)!} \cdot$$
$$\cdot\, \frac{1}{r^{k_2+k_3+\cdots+k_{n-1}+k_n}}\, \frac{1}{w_2^{k_2} w_3^{k_3} \cdots w_{n-1}^{k_{n-1}} w_n^{k_n}},$$

we keep the same name A after eliminating the factor $\widetilde{I}_0$ on the first line, and we reformulate our goal as a more precise

Problem 4.3. *For some specific choice of a fixed constant $r \geqslant 3$, to show that for any $n \geqslant 2$, the coefficient of the constant monomial $w_2^0 \cdots w_n^0$ in the product $C(w) \cdot A(w)$ is at least equal to 1, namely:*

$$1 \;\leqslant\; \big[w_2^0 \cdots w_n^0\big]\Big(C(w_2, \ldots, w_n) \cdot A(w_2, \ldots, w_n)\Big),$$

where:

$$C(w_2, \ldots, w_n) \;:=\; \begin{array}{ccccc} \frac{1-w_2}{1-2\,w_2} & \frac{1-w_2w_3}{1-2\,w_2w_3} & \cdots\cdots\cdots\cdots & \frac{1-w_2w_3\cdots w_n}{1-2\,w_2w_3\cdots w_n} \\ & \frac{1-w_3}{1-2\,w_3+w_2w_3} & \cdots\cdots & \frac{1-w_3\cdots w_n}{1-2\,w_3\cdots w_n+w_2w_3\cdots w_n} \\ & & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & \frac{1-w_n}{1-2\,w_n+w_{n-1}w_n} \end{array},$$

and where:

$$A(w_2, \ldots, w_n) \;:=\; \sum_{\substack{0\leqslant k_2\leqslant n\\ 0\leqslant k_3\leqslant n+k_2\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ 0\leqslant k_{n-1}\leqslant n+k_{n-2}\\ 0\leqslant k_n \;\leqslant n+k_{n-1}}} \frac{n!}{(n-k_2)!}\, \frac{n!}{(n+k_2-k_3)!} \cdots \frac{n!}{(n+k_{n-2}-k_{n-1})!}\, \frac{n!}{(n+k_{n-1}-k_n)!(n+k_n)!} \cdot$$
$$\cdot\, \frac{1}{r^{k_2+k_3+\cdots+k_{n-1}+k_n}}\, \frac{1}{w_2^{k_2} w_3^{k_3} \cdots w_{n-1}^{k_{n-1}} w_n^{k_n}}.$$

Of course, under the hypothesis that the power series expansion of $C(w)$ is known:

$$C(w_2,\dots,w_n) \,=\, \sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} C_{k_2,\dots,k_n}\,(w_2)^{k_2}\cdots(w_n)^{k_n},$$

the coefficient in question writes up as the sum:

$$\boldsymbol{CA}^n_{n-1} \,:=\, \sum_{\substack{0\leqslant k_2\leqslant n\\ 0\leqslant k_3\leqslant n+k_2\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ 0\leqslant k_{n-1}\leqslant n+k_{n-2}\\ 0\leqslant k_n\ \ \leqslant n+k_{n-1}}} \frac{n!}{(n-k_2)!}\,\frac{n!}{(n+k_2-k_3)!}\,\cdots\,\frac{n!}{(n+k_{n-2}-k_{n-1})!}\,\frac{n!}{(n+k_{n-1}-k_n)!(n+k_n)!}\cdot$$

$$\cdot\,\frac{1}{r^{k_2+k_3+\cdots+k_{n-1}+k_n}}\,C_{k_2,k_3,\dots,k_{n-1},k_n},$$

which should satisfy:

$$\boldsymbol{CA}^n_{n-1} \,\overset{?}{\geqslant}\, 1 \qquad (\forall\, n\geqslant 2).$$

5. Approximations of multinomial quotients $M^n_{k_2,\dots,k_n}$

Let us attribute a name to the quotients of multinomial coefficients which have appeared above:

$$M^n_{k_2,k_3,\dots,k_{n-1},k_n} \,:=\, \frac{n!}{(n-k_2)!}\,\frac{n!}{(n+k_2-k_3)!}\,\cdots\cdots\,\frac{n!}{(n+k_{n-1}-k_n)!}\,\frac{n!}{(n+k_n)!}$$

$$=\,\frac{\frac{(n^2)!}{(n-k_2)!\,(n+k_2-k_3)!\cdots(n+k_{n-1}-k_n)!\,(n+k_n)!}}{\frac{(n^2)!}{n!\,n!\cdots n!\,n!}}.$$

When $k_2=k_3=\cdots=k_{n-1}=k_n=0$, this is just:

$$M^n_{0,0,\dots,0,0} \,=\, 1.$$

Lemma 5.1. *For all indices $(k_2,k_3,\dots,k_{n-1},k_n)\neq(0,0,\dots,0,0)$ in the domain:*

$$\begin{aligned}
0 \,&\leqslant\, k_2 \,\leqslant\, n,\\
0 \,&\leqslant\, k_3 \,\leqslant\, n+k_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
0 \,&\leqslant\, k_{n-1} \,\leqslant\, n+k_{n-2},\\
0 \,&\leqslant\, k_n \quad\leqslant\, n+k_{n-1},
\end{aligned}$$

there are strict inequalities:

$$\big(0\,\leqslant\,\big)\qquad M^n_{k_2,k_3,\dots,k_{n-1},k_n} \,<\, 1,$$

with equality $=1$ only when $k_2=k_3=\cdots=k_{n-1}=k_n=0$.

Proof. Coming back to the old (nonnegative) indices:

$$\begin{aligned}
i_n \,&=\, n-k_2,\\
i_{n-1} \,&=\, n+k_2-k_3,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
i_2 \,&=\, n+k_{n-1}-k_n,\\
i_1 \,&=\, n+k_n,
\end{aligned}$$

which satisfy $i_1 + i_2 + \cdots + i_{n-1} + i_n = n\,n = n^2$ and are not all equal to n — otherwise all $k_\lambda = 0$ —, we have to explain the inequalities:

$$\frac{n!}{i_1!}\,\frac{n!}{i_2!}\;\cdots\cdots\;\frac{n!}{i_{n-1}!}\,\frac{n!}{i_n!}\;\overset{?}{<}\;1.$$

After a reordering, we can assume that:

$$i_1 \neq n,\;\;\ldots\ldots,\;\;i_\kappa \neq n,\;\;i_{\kappa+1} = n,\;\;\ldots\ldots,\;\;i_n = n,$$

for a certain integer $1 \leqslant \kappa \leqslant n$. Since the factors $\frac{n!}{n!} = 1$ have no effect, we are led to ask whether:

$$\frac{n!}{i_1!}\;\cdots\cdots\;\frac{n!}{i_\lambda!}\;\cdots\cdots\;\frac{n!}{i_\kappa!}\;\overset{?}{<}\;1.$$

Observing that:

$$i_1 + \cdots + i_\lambda + \cdots + i_\kappa \;=\; \kappa\,n,$$

let us distinguish two cases about these i_λ for every $1 \leqslant \lambda \leqslant \kappa$:

$$i_\lambda \;<\; n \qquad\qquad \text{or} \qquad\qquad i_\lambda \;>\; n.$$

When $i_\lambda < n$, we simplify:

$$\frac{n!}{i_\lambda!} \;=\; n\,\big(n-1\big)\cdots\big(i_\lambda+1\big),$$

and when $i_\lambda > n$, we simplify:

$$\frac{n!}{i_\lambda!} \;=\; \frac{1}{i_\lambda\,(i_\lambda-1)\cdots(n+1)},$$

so that:

$$\frac{n!}{i_1!}\;\cdots\cdots\;\frac{n!}{i_\lambda!}\;\cdots\cdots\;\frac{n!}{i_\kappa!} \;=\; \frac{\prod_{i_\lambda<n}\,n\,(n-1)\,\cdots\,(i_\lambda+1)}{\prod_{i_\lambda>n}\,i_\lambda(i_\lambda-1)\cdots(n+1)}.$$

Now, we observe that in this fraction the number of integer factors at numerator place is equal to the number of integer factors at denominateur place, because the equality above:

$$\kappa\,n \;=\; \sum_{1\leqslant\lambda\leqslant\kappa} i_\lambda \;=\; \sum_{i_\lambda<n} i_\lambda + \sum_{i_\lambda>n} i_\lambda$$

can be rewritten as:

$$\sum_{i_\lambda<n}\,\big(n-i_\lambda\big) \;=\; \sum_{i_\lambda>n}\,\big(i_\lambda-n\big).$$

But *each* integer factor at denominator place is *larger* than *all* integer factors at numerator place, so the fraction must be < 1. □

Visibly, in the quantity under study:

$$M^n_{k_2,k_3,\ldots,k_{n-1},k_n} \;=\; \frac{n!}{(n-k_2)!}\;\frac{n!}{(n+k_2-k_3)!}\;\cdots\cdots\;\frac{n!}{(n+k_{n-1}-k_n)!}\;\frac{n!}{(n+k_n)!},$$

there are two types of quotients:

$$\frac{n!}{(n-k)!}\quad\text{with}\;\;k\geqslant 0 \qquad\qquad \text{and} \qquad\qquad \frac{n!}{(n+\ell)!}\quad\text{with}\;\;\ell\geqslant 0.$$

We can simplify, factorize, and rewrite the first type quotients as:

$$\frac{n!}{(n-k)!} = \frac{n\,(n-1)\cdots(n-k+1)}{1} = n^k \left(1-\frac{0}{n}\right)\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$
$$= n^k \prod_{0\leqslant i\leqslant k-1} \left(1-\frac{i}{n}\right),$$

and the second type quotients as:

$$\frac{n!}{(n+\ell)!} = \frac{1}{(n+\ell)\cdots(n+1)} = \frac{1}{\left(1+\frac{\ell}{n}\right)\cdots\left(1+\frac{1}{n}\right)n^\ell}$$
$$= n^{-\ell} \prod_{1\leqslant j\leqslant \ell} \left(1+\frac{j}{n}\right)^{-1}.$$

In order to estimate the proximity to 1 of these products, let us take their logarithms:

$$\log \prod_{0\leqslant i\leqslant k-1} \left(1-\frac{i}{n}\right) = \log\left(1-\frac{0}{n}\right) + \log\left(1-\frac{1}{n}\right) + \cdots + \log\left(1-\frac{k-1}{n}\right)$$
$$\leqslant 0,$$

and:

$$\log \prod_{1\leqslant j\leqslant \ell} \left(1+\frac{j}{n}\right)^{-1} = -\log\left(1+\frac{1}{n}\right) - \log\left(1+\frac{2}{n}\right) - \cdots - \log\left(1+\frac{\ell}{n}\right)$$
$$\leqslant 0.$$

Certainly, we have already seen implicitly in the proof of the previous Lemma 5.1 that all the logarithms of these products are $\leqslant 0$. But we are now searching for a *minoration* of these coefficients:

$$M^n_{k_2,\ldots,k_n} \geqslant \text{what?}$$

For a reason that will become transparent just after a preliminary lemma, we will soon restrict ourselves to suppose that:

$$k_2 + k_3 + \cdots + k_{n-1} + k_n \leqslant \sqrt{n}.$$

Lemma 5.2. *For all* $0 \leqslant \delta \leqslant 3/5$*:*

$$\log\left(1-\delta\right) \geqslant -\delta - \delta^2,$$

and for all $\varepsilon \geqslant 0$*:*

$$-\log\left(1+\varepsilon\right) \geqslant -\varepsilon.$$

Proof. The first inequality — which is in fact true for $0 \leqslant \delta \leqslant 0,683$ as can be seen with the help of a computer —:

$$-\delta - \frac{\delta^2}{2} - \frac{\delta^3}{3} - \frac{\delta^4}{4} - \frac{\delta^5}{5} - \cdots \overset{?}{\geqslant} -\delta - \delta^2,$$

is equivalent to:

$$\frac{\delta^2}{2} \overset{?}{\geqslant} \frac{\delta^3}{3} + \frac{\delta^4}{4} + \frac{\delta^5}{5} + \cdots.$$

In this inequality under questioning, let us insert a computable infinite sum:

$$1 \overset{?}{\geqslant} \frac{2}{3}\,\delta\left(1+\delta+\delta^2+\cdots\right) \geqslant \frac{2}{3}\,\delta + \frac{2}{4}\,\delta^2 + \frac{2}{5}\,\delta^3 + \cdots,$$

in order to come to an elementary minoration:

$$1 \overset{?}{\geqslant} \frac{2}{3}\,\delta\,\frac{1}{1-\delta} \quad\Longleftrightarrow\quad 3-3\,\delta \overset{\text{yes}}{\geqslant} 2\,\delta.$$

The second inequality $\log(1+\varepsilon) \leqslant \varepsilon$ is well known. □

Now, let us suppose that:

$$k \leqslant \sqrt{n},$$

whence as soon as $n \geqslant 4$:

$$\frac{k-1}{n} < \frac{1}{\sqrt{n}} \leqslant \frac{1}{2} < \frac{3}{5}.$$

Then:

$$\begin{aligned}\sum_{i=0}^{k-1} \log\left(1-\frac{i}{n}\right) &\geqslant -\sum_{i=0}^{k-1}\frac{i}{n} - \sum_{i=0}^{k-1}\frac{i^2}{n^2}\\ &= -\frac{(k-1)\,k}{2\,n} - \frac{(k-1)\,k\,(2\,k-1)}{6\,n^2}\\ &= -\frac{k^2}{2\,n} + \underline{\frac{k}{2\,n}} - \frac{k^3}{3\,n^2} + \underline{\frac{k^2}{2\,n^2}} - \underline{\frac{k}{6\,n^2}}.\end{aligned}$$

The three terms here underlined have a positive contribution and we can even neglect the second of them:

$$\frac{k}{2\,n} + \underline{\frac{k^2}{2\,n^2}} - \frac{k}{6\,n^2} \;\geqslant\; \frac{k}{2\,n}\left(1-\frac{1}{3\,n}\right) \;>\; 0.$$

Therefore, we obtain a useful minoration:

$$\begin{aligned}\sum_{i=0}^{k-1} \log\left(1-\frac{i}{n}\right) \;\geqslant\; -\frac{k^2}{2\,n} - \frac{k^3}{3\,n^2} &= -\frac{k^2}{2\,n} - \frac{k^2}{3\,n}\,\frac{k}{n}\\ [\tfrac{k}{n}\leqslant 1] \qquad &\geqslant -\frac{k^2}{2\,n} - \frac{k^2}{3\,n}\\ &\geqslant -\frac{k^2}{n}.\end{aligned}$$

Next, for the quotients of second type which are present in the various $M^n_{k_2,\ldots,k_n}$, the minoration work is easier:

$$\begin{aligned}-\sum_{j=1}^{\ell} \log\left(1+\frac{j}{n}\right) &\geqslant -\sum_{j=1}^{\ell}\frac{j}{n}\\ &= -\frac{\ell^2}{2\,n} - \frac{\ell}{2\,n}\\ [\ell\leqslant\ell^2] \qquad &\geqslant -\frac{\ell^2}{n}.\end{aligned}$$

Without forgetting the powers n^k and $n^{-\ell}$, these estimates can now be summarized as the following

Lemma 5.3. *For all* $0 \leqslant k \leqslant \frac{3}{5}\, n$*:*

$$\frac{n!}{(n-k)!} \ \geqslant \ n^k \, e^{-\frac{k^2}{n}}$$

and for all $0 \leqslant \ell$*:*

$$\frac{n!}{(n+\ell)!} \ \geqslant \ n^{-\ell} \, e^{-\frac{\ell^2}{n}}. \qquad \square$$

Importantly, we point out that there is a *uniform* minoration:

$$\frac{n!}{(n+m)!} \ \geqslant \ n^{-m} \, e^{-\frac{m^2}{n}},$$

valid for all integers $m \in \mathbb{Z}$, positive or negative, in the range:

$$-\tfrac{3}{5}\, n \ \leqslant \ m \ < \ \infty.$$

Notice that the exponential factor is always $\leqslant 1$.

Next, thanks to all this, we will assume from now on that the range of the integers $k_2, \ldots, k_n$ is restricted to:

$$0 \ \leqslant \ k_2 + k_3 + \cdots + k_{n-1} + k_n \ \leqslant \ \frac{\sqrt{n}}{c(n)},$$

for some function $c(n) \underset{n\to\infty}{\longrightarrow} \infty$ that will be chosen later — think for instance $c(n) := \mathsf{log\,log\,log}\, n$. In particular, this implies that:

$$0 \ \leqslant \ k_2 \ \leqslant \ \tfrac{3}{5}\, n, \qquad\qquad \big|k_2 - k_3\big| \ \leqslant \ \tfrac{\sqrt{n}}{c(n)} \ \leqslant \ \tfrac{3}{5}\, n, \ \ldots\ldots, \ \big|k_{n-1} - k_n\big| \ \leqslant \ \tfrac{\sqrt{n}}{c(n)} \ \leqslant \ \tfrac{3}{5}\, n,$$

so that the lemma applies to minorize:

$$\begin{aligned}
M^n_{k_2,k_3,\ldots,k_{n-1},k_n} \ &= \ \frac{n!}{(n-k_2)!} \, \frac{n!}{(n+k_2-k_3)!} \ \cdots\cdots \ \frac{n!}{(n+k_{n-1}-k_n)!} \, \frac{n!}{(n+k_n)!} \\
&\geqslant \ \underline{n^{k_2}}_{\circ} \, e^{-\frac{k_2^2}{n}} \, \underline{n^{-k_2+k_3}}_{\circ} \, e^{-\frac{(k_2-k_3)^2}{n}} \ \cdots\cdots \ \underline{n^{-k_{n-1}+k_n}}_{\circ} \, e^{-\frac{(k_{n-1}-k_n)^2}{n}} \, \underline{n^{-k_n}}_{\circ} \, e^{-\frac{k_n^2}{n}} \\
&= \ e^{-\frac{1}{n}\left[k_2^2+(k_2-k_3)^2+\cdots+(k_{n-1}-k_n)^2+k_n^2\right]} \\
&= \ e^{-\frac{1}{n}\left[2\,k_2^2-2\,k_2k_3+2\,k_3^2+\cdots+2\,k_{n-1}^2-2\,k_{n-1}k_n+2\,k_n^2\right]} \\
&\geqslant \ e^{-\frac{1}{n}\left[2\,(k_2+k_3+\cdots+k_{n-1}+k_n)^2\right]} \\
&\geqslant \ e^{-\frac{1}{n}\,2\,\frac{n}{c(n)^2}} \\
&= \ e^{-\frac{2}{c(n)^2}} \ \underset{n\to\infty}{\longrightarrow} \ 1.
\end{aligned}$$

We thus have proved the key

Proposition 5.4. *For any choice of function* $c(n) \overset{n\to\infty}{\longrightarrow} \infty$*, the quantities:*

$$M^n_{k_2,k_3,\ldots,k_{n-1},k_n} \ = \ \frac{n!}{(n-k_2)!} \, \frac{n!}{(n+k_2-k_3)!} \ \cdots\cdots \ \frac{n!}{(n+k_{n-1}-k_n)!} \, \frac{n!}{(n+k_n)!}$$

enjoy the inequalities:

$$e^{-\frac{2}{c(n)^2}} \ \leqslant \ M^n_{k_2,k_3,\ldots,k_{n-1},k_n} \ \leqslant \ 1,$$

when their indices range in the set:

$$\Big\{\big(k_2,k_3,\dots,k_{n-1},k_n\big)\in\mathbb{N}^n\colon\ \begin{array}{l} 0\ \leqslant\ k_2\ \leqslant\ n,\\ 0\ \leqslant\ k_3\ \leqslant\ n+k_2,\\ \cdots\cdots\cdots\cdots\cdots\\ 0\ \leqslant\ k_{n-1}\ \leqslant\ n+k_{n-2},\\ 0\ \leqslant\ k_n\ \ \leqslant\ n+k_{n-1},\end{array}$$
$$k_2+k_3+\cdots+k_{n-1}+k_n\ \leqslant\ \frac{\sqrt{n}}{c(n)}\Big\}. \qquad \square$$

6. Majorant power series $\widehat{C}(w_2,\dots,w_n)$ and its diagonalization $\widehat{C}(x,\dots,x)$

Now, come back to:

$$F(x,y)\ =\ \frac{1-y}{1-2\,y+x\,y}$$

and observe that for all $3\leqslant i\leqslant n-1$:

$$\frac{1-x^{i-1}}{1-2\,x^{i-1}+x^i}\ =\ F\big(x,\,x^{i-1}\big).$$

Its expansion:

$$\begin{aligned} F(x,y)\ &=\ 1+\frac{y-x\,y}{1-2\,y+x\,y}\\ &=:\ 1+\sum_{\ell=1}^{\infty}y^\ell\sum_{k=0}^{\infty}x^k\,F_{k,\ell}\end{aligned}$$

will be (easily) computed soon. With $F(x,y)$, introduce also — notice the single sign change in the denominator:

$$\begin{aligned}\widehat{F}(x,y)\ =\ \frac{1-y}{1-2\,y-x\,y}\ &=\ 1+\frac{y+x\,y}{1-2\,y-x\,y}\\ &=:\ 1+\sum_{\ell=1}^{\infty}y^\ell\sum_{k=0}^{\infty}x^k\,\widehat{F}_{k,\ell},\end{aligned}$$

a new function which will act as a *majorant series*, in the sense that:

$$\big|F_{k,\ell}\big|\ \leqslant\ \widehat{F}_{k,\ell}\qquad\qquad(\forall\,k\geqslant 0,\,\forall\,\ell\geqslant 0).$$

Such inequalities are made transparent from the following clear explicit expressions, in which just a factor $(-1)^k$ drops.

Lemma 6.1. *With the convention that* $\binom{\ell-1}{-1}=0=\binom{\ell-1}{\ell}$*, the power series expansions are:*

$$F(x,y)\ =\ 1+\sum_{\ell=1}^{\infty}y^\ell\sum_{0\leqslant k\leqslant\ell}(-1)^k\,x^k\Big[2^{\ell-1-k}\,\tbinom{\ell-1}{k}+2^{\ell-k}\,\tbinom{\ell-1}{k-1}\Big],$$

$$\widehat{F}(x,y)\ =\ 1+\sum_{\ell=1}^{\infty}y^\ell\sum_{0\leqslant k\leqslant\ell}\qquad x^k\Big[2^{\ell-1-k}\,\tbinom{\ell-1}{k}+2^{\ell-k}\,\tbinom{\ell-1}{k-1}\Big].$$

Proof. Expand:

$$\begin{aligned} F(x,y) &= 1+\frac{y-x\,y}{1-y\,(2-x)} \\ &= 1+\big(y-x\,y\big)\sum_{h=0}^{\infty} y^h\,\big(2-x\big)^h \\ &= 1+\big(y-x\,y\big)\sum_{h=0}^{\infty} y^h \sum_{0\leqslant m\leqslant h} (-1)^m\,x^m\,2^{h-m}\,\binom{h}{m}. \end{aligned}$$

[The series converge *e.g.* for $|x|,|y|<\frac{1}{4}$ since in the denominator $1-2\,\frac{1}{4}-\frac{1}{4^2}>0$, but we will not need precise information about the domain of convergence.] Two double sums must be reorganized. In the first one, replace $h=\ell-1$ and $m=k$:

$$\sum_{h=0}^{\infty} y^{h+1} \sum_{0\leqslant m\leqslant h} (-1)^m\,x^m\,2^{h-m}\,\binom{h}{m} \;=\; \sum_{\ell=1}^{\infty} y^{\ell} \sum_{0\leqslant k\leqslant \ell-1} (-1)^k\,x^k\,2^{\ell-1-k}\,\binom{\ell-1}{k},$$

and observe that the last sum can be extended to the range $0\leqslant k\leqslant \ell$, thanks to the convention. In the second one, replace $h=\ell-1$ and $m=k-1$:

$$-\sum_{h=0}^{\infty} y^{h+1} \sum_{0\leqslant m\leqslant h} (-1)^m\,x^{m+1}\,2^{h-m}\,\binom{h}{m} \;=\; -\sum_{\ell=1}^{\infty} y^{\ell} \sum_{1\leqslant k\leqslant \ell} (-1)^{k-1}\,x^k\,2^{\ell-k}\,\binom{\ell-1}{k-1},$$

and observe that the term $k=0$ in the sum can be included, thanks to the convention. Adding these two expressions yield the stated power expansion of $F(x,y)$.

Next, for what concerns:

$$\begin{aligned} \widehat{F}(x,y) &= 1+\frac{y+x\,y}{1-2\,y-x\,y} \\ &= 1+\big(y+x\,y\big)\sum_{h=0}^{\infty} y^h\,\big(2+x\big)^h \\ &= 1+\big(y+x\,y\big)\sum_{h=0}^{\infty} y^h \sum_{0\leqslant m\leqslant h} x^m\,2^{h-m}\,\binom{h}{m}, \end{aligned}$$

exactly the same transformations work, except that the $(-1)^m$ factor has disappeared. □

Next, our goal is to introduce a majorant power series $\widehat{C}(w_2,\ldots,w_n)$ for the power series $C(w_2,\ldots,w_n)$. As anticipated above, it is now clear by means of the triangle inequality that:

$$\big|F_{k,\ell}\big| \;\leqslant\; \widehat{F}_{k,\ell}, \tag{6.2}$$

for all $k\geqslant 0$ and all $\ell\geqslant 0$. In terms of $F(x,y)$ and of the already seen power series:

$$E(x) \;:=\; \frac{1-x}{1-2\,x} \;=\; \sum_{k=0}^{\infty} E_k\,x^k,$$

having positive coefficients $E_0 = 1$ and $E_k = 2^{k-1}$ for $k \geqslant 1$, recall from (4.1) that:

$$C\big(w_2,\dots,w_n\big) \;=:\; \prod_{2\leqslant i\leqslant n} E\big(w_2\cdots w_i\big) \prod_{2\leqslant i<j\leqslant n} F\big(w_i,\; w_{i+1}\cdots w_j\big).$$

Hence we may introduce similarly:

$$\widehat{C}\big(w_2,\dots,w_n\big) \;:=\; \prod_{2\leqslant i\leqslant n} E\big(w_2\cdots w_i\big) \prod_{2\leqslant i<j\leqslant n} \widehat{F}\big(w_i,\; w_{i+1}\cdots w_j\big).$$

The expansions of the factors of the first product show as:

$$\begin{aligned} C\big(w_2,\dots,w_n\big) \;&=\; \prod_{2\leqslant i\leqslant n} \Big(\sum_{k=0}^{\infty} E_k\,\big(w_2\cdots w_i\big)^k\Big) \\ &\qquad \prod_{2\leqslant i<j\leqslant n} \Big(\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} F_{k,\ell}\,\big(w_i\big)^k\,\big(w_{i+1}\cdots w_j\big)^\ell\Big) \\ &=:\; \sum_{k_2,\dots,k_n\,\geqslant\,0} C_{k_2,\dots,k_n}\,\big(w_2\big)^{k_2}\cdots\big(w_n\big)^{k_n}, \end{aligned}$$

and similarly:

$$\begin{aligned} \widehat{C}\big(w_2,\dots,w_n\big) \;&=\; \prod_{2\leqslant i\leqslant n} \Big(\sum_{k=0}^{\infty} E_k\,\big(w_2\cdots w_i\big)^k\Big) \\ &\qquad \prod_{2\leqslant i<j\leqslant n} \Big(\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \widehat{F}_{k,\ell}\,\big(w_i\big)^k\,\big(w_{i+1}\cdots w_j\big)^\ell\Big) \\ &=:\; \sum_{k_2,\dots,k_n\,\geqslant\,0} \widehat{C}_{k_2,\dots,k_n}\,\big(w_2\big)^{k_2}\cdots\big(w_n\big)^{k_n}. \end{aligned}$$

Since all $E_k \geqslant 0$ and all $\widehat{F}_{k,\ell} \geqslant 0$, we have all $\widehat{C}_{k_2,\dots,k_n} \geqslant 0$ as well — however, many $C_{k_2,\dots,k_n}$ are $\leqslant -1$.

Thanks to (6.2) and to the triangle inequality in expansions, we obtain:

$$\big|C_{k_2,\dots,k_n}\big| \;\leqslant\; \widehat{C}_{k_2,\dots,k_n}, \tag{6.3}$$

for all $k_2,\dots,k_n \geqslant 0$, which means that $\widehat{C}$ is a *majorant power series* for C. Notice that:

$$C_{k_2,\dots,k_n} \;\in\; \mathbb{Z} \qquad\qquad \text{and} \qquad\qquad \widehat{C}_{k_2,\dots,k_n} \;\in\; \mathbb{N}.$$

Now, passing to the diagonal:

$$\big\{w_2 \;=\; \cdots \;=\; w_n \;=:\; x\big\},$$

we deduce for every $k \in \mathbb{N}$, again by means of the triangle inequality:

$$\begin{aligned} \big|C_h\big| \ &= \ \Big| \sum_{k_2+\cdots+k_n=h} C_{k_2,\ldots,k_n} \Big| \\ &\leqslant \ \sum_{k_2+\cdots+k_n=h} \big|C_{k_2,\ldots,k_n}\big| \\ &\leqslant \ \sum_{k_2+\cdots+k_n=h} \widehat{C}_{k_2,\ldots,k_n} \ =: \ \widehat{C}_h. \end{aligned} \qquad \text{[(6.3)]}$$

In fact, these integers $\widehat{C}_h \geqslant 0$ express as *coefficients* of the diagonal majorant series:

$$\begin{aligned} \widehat{C}^{n-1}(x) \ &:= \widehat{C}(x,\ldots,x) \\ &= \ \prod_{2\leqslant i\leqslant n} \frac{1-x^{i-1}}{1-2\,x^{i-1}} \prod_{2\leqslant i<j\leqslant n} \frac{1-x^{j-i}}{1-2\,x^{j-i}-x^{j-i+1}} \\ &= \ \sum_{k_2,\ldots,k_n\geqslant 0} \widehat{C}_{k_2,\ldots,k_n}\, x^{k_2}\cdots x^{k_n} \\ &= \ \sum_{h=0}^{\infty} \Big(\sum_{k_2+\cdots+k_n=h} \widehat{C}_{k_2,\ldots,k_n} \Big) x^h \ =: \ \sum_{h=0}^{\infty} \widehat{C}_h\, x^h. \end{aligned}$$

Let us therefore state these observations as a

Lemma 6.4. *The* 1*-variable products/series:*

$$\begin{aligned} C^{n-1}(x) \ &:= \ \prod_{i=1}^{n-1} \frac{1-x^i}{1-2\,x^i} \ \prod_{i=2}^{n-1} \Big(\frac{1-x^{i-1}}{1-2\,x^{i-1}+x^i} \Big)^{n-i} \ = \ \sum_{h=0}^{\infty} C_h^{n-1}\, x^h, \\ \widehat{C}^{n-1}(x) \ &:= \ \prod_{i=1}^{n-1} \frac{1-x^i}{1-2\,x^i} \ \prod_{i=2}^{n-1} \Big(\frac{1-x^{i-1}}{1-2\,x^{i-1}-x^i} \Big)^{n-1} \ = \ \sum_{h=0}^{\infty} \widehat{C}_h^{n-i}\, x^h, \end{aligned}$$

have coefficients satisfying the inequalities:

$$\big|C_h^{n-1}\big| \ \leqslant \ \widehat{C}_h^{n-1} \qquad (\forall\, h\geqslant 0). \qquad \square$$

7. Positivity of diagonal sums coefficients C_h^{n-1}

Now, study the power series $C(w_2,\ldots,w_n)$ along the diagonal:

$$\big\{ w_2 \ = \ \cdots \ = \ w_n \ =: \ x \big\},$$

that is to say, introduce as in Lemma 6.4:

$$\begin{aligned} C^{n-1}(x) \ = \ & \frac{1-x}{1-2\,x} \ \frac{1-x^2}{1-2\,x^2} \ \frac{1-x^3}{1-2\,x^3} \ \cdots\cdots\cdots\cdots\cdots \ \frac{1-x^{n-1}}{1-2\,x^{n-1}} \\ & \Big(\frac{1-x}{1-2\,x+x^2} \Big)^{n-2} \Big(\frac{1-x^2}{1-2\,x^2+x^3} \Big)^{n-3} \cdots\cdots \Big(\frac{1-x^{n-2}}{1-2\,x^{n-2}+x^{n-1}} \Big)^{1}. \end{aligned}$$

Furthermore, on the second line, the first fraction to the power $(\cdot)^{n-2}$ trivially simplifies as:

$$\frac{1-x}{(1-x)^2} \ = \ \frac{1}{1-x},$$

whence:

$$C^{n-1}(x) \;=\; \frac{1-x}{1-2\,x}\;\frac{1-x^2}{1-2\,x^2}\;\frac{1-x^3}{1-2\,x^3}\;\cdots\cdots\cdots\cdots\;\frac{1-x^{n-1}}{1-2\,x^{n-1}}$$
$$\left(\frac{1}{1-x}\right)^{n-2}\left(\frac{1-x^2}{1-2\,x^2+x^3}\right)^{n-3}\cdots\cdots\left(\frac{1-x^{n-2}}{1-2\,x^{n-2}+x^{n-1}}\right)^{1}.$$

Let us focus on the second line, which we now call:

$$P^{n-1}(x) \;:=\; \left(\frac{1}{1-x}\right)^{n-2}\left(\frac{1-x^2}{1-2\,x^2+x^3}\right)^{n-3}\cdots\cdots\left(\frac{1-x^{n-2}}{1-2\,x^{n-2}+x^{n-1}}\right)^{1}$$
$$=:\; \sum_{h=0}^{\infty} P_h^{n-1}\,x^h.$$

We believe that all the coefficients of the full product $C^{n-1}(x)$ are positive, but a restricted statement will be enough for our purposes.

Lemma 7.1. *For all indices h in the range:*

$$0 \;\leqslant\; h \;\leqslant\; \lfloor\sqrt{n}\rfloor$$

one has:

$$P_h^{n-1} \;\geqslant\; 1,$$
$$C_h^{n-1} \;\geqslant\; 2^h.$$

Proof. First, we make the following transformation for each term in the product $P^{n-1}(x)$:

$$\left(\frac{1-x^k}{1-2\,x^k+x^{k+1}}\right)^{n-k-1} = \left(\frac{1-x^k}{1-x^k-(x^k-x^{k+1})}\right)^{n-k-1}$$
$$= \left(\frac{1}{1-\frac{x^k-x^{k+1}}{1-x^k}}\right)^{n-k-1}$$
$$= \left(\frac{1}{1-\frac{x^k}{1+x+\cdots+x^{k-1}}}\right)^{n-k-1} \qquad (1\leqslant k\leqslant n-2).$$

Using the expansion and factorization (valid for $|T|<1$):

$$\frac{1}{1-T} = \Big(1+T\Big)\sum_{i=0}^{\infty} T^{2i} = \Big(1+T\Big)\Big(1+T^2+T^4+T^6+\cdots\Big)$$

and substituting $T=\frac{x^k}{1+x+\cdots+x^{k-1}}$ (with $|x|$ so small that $|T|<1$ for any $1\leqslant k\leqslant n-2$) gives us:

$$\left(\frac{1}{1-\frac{x^k}{1+x+\cdots+x^{k-1}}}\right)^{1} = \left(1+\frac{x^k}{1+x+\cdots+x^{k-1}}\right)\left(\sum_{i=0}^{\infty}\Big(\frac{x^k}{1+x+\cdots+x^{k-1}}\Big)^{2i}\right)$$
$$= \left(\frac{1+x+\cdots+x^k}{1+x+\cdots+x^{k-1}}\right)\left(\sum_{i=0}^{\infty}\Big(\frac{x^k}{1+x+\cdots+x^{k-1}}\Big)^{2i}\right).$$

We then put together these expansions of terms in the product $P^{n-1}(x)$ to obtain:

$$\begin{aligned}
P^{n-1}(x) =& \Big(1+x\Big)^{n-2}\Big(1+x^2+x^4+x^6+\cdots\Big)^{n-2}\\
&\cdot\Big(\frac{1+x+x^2}{1+x}\Big)^{n-3}\Big(1+\Big(\frac{x^2}{1+x}\Big)^2+\Big(\frac{x^2}{1+x}\Big)^4+\cdots\Big)^{n-3}\\
&\qquad\qquad\cdots\cdots\cdots\\
&\cdot\Big(\frac{1+x+\cdots+x^{k-1}+x^k}{1+x+\cdots+x^{k-1}}\Big)^{n-k-1}\Big(\sum_{i=0}^{\infty}\Big(\frac{x^k}{1+x+\cdots+x^{k-1}}\Big)^{2i}\Big)^{n-k-1}\\
&\qquad\qquad\cdots\cdots\cdots\\
&\cdot\Big(\frac{1+x+\cdots+x^{n-3}+x^{n-2}}{1+x+\cdots+x^{n-3}}\Big)^{1}\Big(1+\Big(\frac{x^{n-2}}{1+x+\cdots+x^{n-3}}\Big)^2+\cdots\Big)^{1}.
\end{aligned}$$

Notice that the product of the first terms in all lines admits simplification as follows:

$$\begin{aligned}
&\Big(1+x\Big)^{n-2}\Big(\frac{1+x+x^2}{1+x}\Big)^{n-3}\cdots\Big(\frac{1+\cdots+x^{k-1}+x^k}{1+\cdots+x^{k-1}}\Big)^{n-k-1}\cdots\Big(\frac{1+\cdots+x^{n-3}+x^{n-2}}{1+\cdots+x^{n-3}}\Big)^{1}\\
&\qquad=(1+x)(1+x+x^2)\cdots(1+x+\cdots+x^k)\cdots(1+x+\cdots+x^{n-2}),
\end{aligned}$$

while the other terms can be expanded using

$$\begin{aligned}
\Big(\sum_{i=0}^{\infty}T^{2i}\Big)^m &= \Big(1+T^2+T^4+\cdots+T^{2j}+\cdots\Big)^m\\
&=\sum_{j=0}^{\infty}\binom{m+j-1}{j}T^{2j}\\
&=1+\binom{m}{1}T^2+\binom{m+1}{2}T^4+\cdots+\binom{m+j-1}{j}T^{2j}+\cdots.
\end{aligned}$$

The expansion of $P^{n-1}(x)$ now becomes

$$\begin{aligned}
P^{n-1}(x) =& \Big(1+x\Big)\Big(1+x+x^2\Big)\cdots\Big(1+x+\cdots+x^k\Big)\cdots\Big(1+x+\cdots+x^{n-2}\Big)\\
&\cdot\Big(1+\binom{n-2}{1}x^2+\binom{n-1}{2}x^4+\binom{n}{3}x^6+\cdots\Big)\\
&\cdot\Big(1+\binom{n-3}{1}\Big(\frac{x^2}{1+x}\Big)^2+\binom{n-2}{2}\Big(\frac{x^2}{1+x}\Big)^4+\cdots\Big)\\
&\qquad\qquad\cdots\cdots\cdots\\
&\cdot\Big(\sum_{i=0}^{\infty}\binom{n-k-1+i-1}{i}\Big(\frac{x^k}{1+x+\cdots+x^{k-1}}\Big)^{2i}\Big)\\
&\qquad\qquad\cdots\cdots\cdots\\
&\cdot\Big(1+\Big(\frac{x^{n-2}}{1+x+\cdots+x^{n-3}}\Big)^2+\Big(\frac{x^{n-2}}{1+x+\cdots+x^{n-3}}\Big)^4+\cdots\Big).
\end{aligned}$$

Since we are only interested in the coefficients P_h^{n-1} with $0\leqslant h\leqslant\lfloor\sqrt{n}\rfloor$, we will ignore the terms $\big(\frac{x^k}{1+x+\cdots+x^{k-1}}\big)^{2i}$ with $k\cdot 2i>n$, i.e. with $i>\frac{n}{2k}$. The first $\lfloor\sqrt{n}\rfloor$ coefficients

in the power series expansion of $P^{n-1}(x)$ are the same as those of

$$\Big(1+x\Big)\Big(1+x+x^2\Big)\cdots\Big(1+x+\cdots+x^k\Big)\cdots\Big(1+x+\cdots+x^{n-2}\Big)$$
$$.\Big(1+\binom{n-2}{1}x^2+\binom{n-1}{2}x^4+\cdots+\binom{n-2+\lfloor\frac{\sqrt{n}}{2}\rfloor-1}{\lfloor\frac{\sqrt{n}}{2}\rfloor}x^{2\lfloor\frac{\sqrt{n}}{2}\rfloor}\Big)$$
$$.\Big(1+\binom{n-3}{1}\Big(\frac{x^2}{1+x}\Big)^2+\cdots+\binom{n-3+\lfloor\frac{\sqrt{n}}{4}\rfloor-1}{\lfloor\frac{\sqrt{n}}{4}\rfloor}\Big(\frac{x^2}{1+x}\Big)^{2\lfloor\frac{\sqrt{n}}{4}\rfloor}\Big)$$
$$\cdots\cdots\cdots$$
$$.\Big(1+\cdots+\binom{n-k-1+\lfloor\frac{\sqrt{n}}{2k}\rfloor-1}{\lfloor\frac{\sqrt{n}}{2k}\rfloor}\Big(\frac{x^k}{1+x+\cdots+x^{k-1}}\Big)^{2\lfloor\frac{\sqrt{n}}{2k}\rfloor}\Big)$$
$$\cdots\cdots\cdots$$
$$.\Big(1+\binom{n-\lfloor\frac{\sqrt{n}}{2}\rfloor-1}{1}\Big(\frac{x^{\lfloor\frac{\sqrt{n}}{2}\rfloor}}{1+x+\cdots+x^{\lfloor\frac{\sqrt{n}}{2}\rfloor-1}}\Big)^2\Big).$$

Now it is clear that in order to show the positivity of P_h^{n-1} for all $0\leqslant h\leqslant\lfloor\sqrt{n}\rfloor$, it suffices to prove that the product

$$\Big(1+x\Big)^{2\lfloor\frac{\sqrt{n}}{4}\rfloor}\Big(1+x+x^2\Big)^{2\lfloor\frac{\sqrt{n}}{6}\rfloor}\cdots\Big(1+x+\cdots+x^{\lfloor\frac{\sqrt{n}}{2}\rfloor-1}\Big)^2$$

is divisible by

$$\Big(1+x\Big)\Big(1+x+x^2\Big)\cdots\Big(1+x+\cdots+x^k\Big)\cdots\Big(1+x+\cdots+x^{n-2}\Big),$$

and at the same time that the quotient also has nonnegative coefficients.

Note that for any $j\geqslant 0$, one has

$$1+x+\cdots+x^{kj+k-1}=\Big(1+x+\cdots+x^{k-1}\Big)\Big(1+x^k+x^{2k}+\cdots+x^{kj}\Big),$$

that is $1+x+\cdots+x^{k-1}$ is divisible by $1+x+\cdots+x^{kj+k-1}$ with quotient having nonnegative coefficients.

Now, we divide the set of indices $\Big\{1,2,\ldots,(\lfloor\sqrt{n}\rfloor)^2-1\Big\}$ into $\lfloor\sqrt{n}\rfloor$ disjoint sets:

$$\Big\{\lfloor\sqrt{n}\rfloor j+1,\lfloor\sqrt{n}\rfloor j+2,\ldots,\lfloor\sqrt{n}\rfloor j+\lfloor\sqrt{n}\rfloor\Big\}$$

for $j=0,1,\ldots,\lfloor\sqrt{n}\rfloor-1$. Then, for each index k, the number of integers of the form $kj+k-1$ in the interval $\Big\{\lfloor\sqrt{n}\rfloor k+1,\lfloor\sqrt{n}\rfloor k+2,\ldots,\lfloor\sqrt{n}\rfloor k+\lfloor\sqrt{n}\rfloor\Big\}$ is at least $\frac{\lfloor\sqrt{n}\rfloor}{k}$.

Since $2\lfloor\frac{\sqrt{n}}{2k}\rfloor\leqslant\frac{\lfloor\sqrt{n}\rfloor}{k}$, the polynomial $\Big(1+x+\cdots+x^{k-1}\Big)^{2\lfloor\frac{\sqrt{n}}{2k}\rfloor}$ is divisible by

$$\prod_{i=\lfloor\sqrt{n}\rfloor k+1}^{\lfloor\sqrt{n}\rfloor(k+1)}\Big(1+x+\cdots+x^i\Big),$$

with quotient having nonnegative coefficients. Taking in account all the values of $k=1,2,\ldots,\lfloor\frac{\sqrt{n}}{2}\rfloor-1$, and making the product of all the $\lfloor\frac{\sqrt{n}}{2}\rfloor-1$ terms gives us the desired divisibility.

At this point, notice further that the set $\Big\{1, 2, \ldots, \lfloor\sqrt{n}\rfloor\Big\}$, corresponding to $k = 0$, has not been used in obtaining the above divisibility. Thus, the first $\lfloor\sqrt{n}\rfloor$ coefficients of $P^{n-1}(x)$ are those of the product between

$$\Big(1+x\Big)\Big(1+x+x^2\Big)\cdots\Big(1+x+\cdots+x^{\lfloor\sqrt{n}\rfloor}\Big)$$

and a power series having constant coefficient 1 and the first $\lfloor\sqrt{n}\rfloor$ coefficients nonnegative. This clearly implies the positivity of P_h^{n-1} for all $0 \leqslant h \leqslant \lfloor\sqrt{n}\rfloor$.

For the first $\lfloor\sqrt{n}\rfloor$ coefficients in the power series expansion of $C^{n-1}(x)$, it is enough to consider the product

$$\left(\frac{1-x}{1-2x}\right)P^{n-1}(x),$$

since all the remaining terms in the product $C^{n-1}(x)$ have power series expansions with nonnegative coefficients and constant coefficient 1. Now using the expansion

$$\frac{1-x}{1-2x} = 1 + \sum_{i=1}^{\infty} 2^{i-1}x^i,$$

we get

$$C_h^{n-1} = P_h^{n-1} + \sum_{i=1}^{h} 2^{i-1}P_{h-i}^{n-1}.$$

Since we have already showed that $P_h^{n-1} \geqslant 1$ for all $0 \leqslant h \leqslant \lfloor\sqrt{n}\rfloor$, it follows that

$$C_h^{n-1} \geqslant 1 + \sum_{i=1}^{h} 2^{i-1} = 2^h$$

for all $0 \leqslant h \leqslant \lfloor\sqrt{n}\rfloor$. This finishes our proof of the lemma. □

8. Cauchy inequalities

Next, we will set up a useful (and trivial) version of the Cauchy inequalities for power series having nonnegative coefficients. We start by determining the radius of convergence $\mathrm{R} > 0$ of $C^{n-1}(x)$ and the one $\widehat{\mathrm{R}} > 0$ of $\widehat{C}^{n-1}(x)$, the two products of Lemma 6.4.

Lemma 8.1. *The smallest moduli of poles of the two rational functions $C^{n-1}(x)$ and $\widehat{C}^{n-1}(x)$ are:*

$$\mathrm{R} := \tfrac{1}{2} = 0.5 \qquad \textit{and} \qquad \widehat{\mathrm{R}} := \sqrt{2}-1 \approx 0.414\cdots.$$

Proof. The moduli of the roots of the denominator of the first product $\prod_{1\leqslant i\leqslant n-1}^{*} \frac{}{1-2\,x^i}$ appearing in $C^{n-1}(x)$ are $\frac{1}{2}$, $\frac{1}{\sqrt[2]{2}}$, $\frac{1}{\sqrt[3]{2}}$, $\ldots$, $\frac{1}{\sqrt[n-1]{2}}$, and the smallest among them is $\frac{1}{2}$. But then in the disc $\Big\{x \in \mathbb{C}\colon\ |x| < \frac{1}{2}\Big\}$, we assert that all denominators in the second product constituting $C^{n-1}(x)$ are nowhere vanishing. Indeed, as already observed above, taking account of the simplification for $i = 2$:

$$\frac{1-x^{2-1}}{1-2\,x^{2-1}+x^2} = \frac{1}{1-x},$$

this second product writes as:

$$\Big(\frac{1}{1-x}\Big)^{n-2}\prod_{i=3}^{n-1}\Big(\frac{*}{1-2\,x^{i-1}+x^i}\Big)^{n-i}.$$

Then the root 1 is certainly $>\frac{1}{2}$, while the subsequent denominators for $3\leqslant i\leqslant n$ are nonvanishing when $|x|\leqslant\frac{1}{2}$, because:

$$\begin{aligned}\big|1-2\,x^{i-1}+x^i\big| &\geqslant 1-2\,|x|^{i-1}-|x|^i\\ &\geqslant 1-2\,\big(\tfrac{1}{2}\big)^{i-1}-\big(\tfrac{1}{2}\big)^i\\ &\geqslant 1-2\,\big(\tfrac{1}{2}\big)^{2}-\big(\tfrac{1}{2}\big)^3\ =\ \tfrac{3}{8}\ >\ 0.\end{aligned}$$

On the other hand, while the first product constituting $\widehat{C}^{n-1}(x)$ is exactly the same, such a simplification in the second product does not occur, and in fact, in:

$$\Big(\frac{*}{1-2\,x-x^2}\Big)^{n-2}\prod_{i=3}^{n-1}\Big(\frac{*}{1-2\,x^{i-1}-x^i}\Big)^{n-i},$$

the same minoration for $3\leqslant i\leqslant n-1$ applies:

$$\big|1-2\,x^{i-1}-x^i\big|\ \geqslant\ 1-2\,|x|^{i-1}-|x|^i\ \geqslant\ \tfrac{3}{8},$$

whereas the positive root $\sqrt{2}-1$ of $1-2\,x-x^2=0$ is smaller than $\frac{1}{2}$, and the other root $-1-\sqrt{2}$ has (much) larger modulus. □

Let therefore $0<\rho<\sqrt{2}-1$ be any radius in these convergence discs. A trivial version of the Cauchy inequalities for power series having nonnegative coefficients is as follows. Using the notation of Section **6**, from:

$$\widehat{C}^{n-1}(\rho)\ =\ \sum_{h=0}^{\infty}\widehat{C}_h^{n-1}\,\rho^h,$$

it comes for any $h\in\mathbb{N}$ fixed, since all terms are $\geqslant 0$:

$$\widehat{C}^{n-1}(\rho)\ \geqslant\ \widehat{C}_h^{n-1}\,\rho^h.$$

Soon, we will take $\rho=\rho(n)\ \xrightarrow[n\to\infty]{>}\ 0$, in fact:

$$\rho\ :=\ \frac{1}{\sqrt{n}}\qquad\text{(later)}.$$

Observation 8.2. *For any* $0<\rho<\sqrt{2}-1$ *and every* $h\in\mathbb{N}$*:*

$$\widehat{C}_h\ \leqslant\ \frac{1}{\rho^h}\,\widehat{C}(\rho).$$ □

Section **11** provides an exploration of the way moduli of the elementary constituents $\frac{1-x^k}{1-2\,x^k}$ and $\frac{1-x^\ell}{1-2\,x^\ell-x^{\ell+1}}$ vary with drastic oscillations on circles $\{|x|=\rho\}$.

Thanks to these basic Cauchy inequalities, we can now start to control the growth of $\widehat{C}^{n-1}(\rho)$.

9. Estimations of $\widehat{C}(\frac{1}{r})$ and of $C(\frac{1}{r})$

At first, we reorganize $\widehat{C}^{n-1}(x)$ from Lemma 6.4, writing its second product up to $i=n$ included instead of $i=n-1$, using $(*)^{n-n}=1$:

$$\begin{aligned}
\widehat{C}^{n-1}(x) &= \prod_{i=1}^{n-1} \frac{1-x^i}{1-2\,x^i} \prod_{i=2}^{n} \Big(\frac{1-x^{i-1}}{1-2\,x^{i-1}-x^i}\Big)^{n-i} \\
&= \underbrace{\prod_{i=1}^{n-1} \frac{1}{1-2\,x^i}}_{i=:\,k} \underbrace{\prod_{i=1}^{n-1} \big(1-x^i\big)}_{i=:\,k} \underbrace{\prod_{i=2}^{n} \big(1-x^{i-1}\big)^{n-i}}_{i=:\,k+1} \underbrace{\prod_{i=2}^{n} \frac{1}{(1-2\,x^{i-1}-x^i)^{n-i}}}_{i=:\,k+1} \\
&= \prod_{k=1}^{n-1} \frac{1}{1-2\,x^k} \underline{\prod_{k=1}^{n-1} \big(1-x^k\big)^1 \prod_{k=1}^{n-1} \big(1-x^k\big)^{n-k-1}} \prod_{k=1}^{n-1} \frac{1}{(1-2\,x^k-x^{k+1})^{n-k-1}} \\
&= \prod_{k=1}^{n-1} \frac{1}{1-2\,x^k} \underline{\prod_{k=1}^{n-1} \big(1-x^k\big)^{n-k}} \prod_{k=1}^{n-1} \frac{1}{(1-2\,x^k-x^{k+1})^{n-k-1}}.
\end{aligned}$$

In order to set up a general statement, we will take:

$$x \,:=\, \frac{1}{r},$$

with $r=r(n) \underset{n\to\infty}{\longrightarrow} \infty$, always with $0<\frac{1}{r}<\sqrt{2}-1$. In fact, to fix ideas, we shall assume at least $r \geqslant 10$.

Lemma 9.1. *One has:*

$$\widehat{C}\big(\tfrac{1}{r}\big) \;\leqslant\; e^{\frac{n}{r}+12\,\frac{n}{r^2}}.$$

Proof. Take logarithm:

$$\begin{aligned}
\log \widehat{C}\big(\tfrac{1}{r}\big) &= \sum_{k=1}^{n-1} \Big(-\log\Big(1-\frac{2}{r^k}\Big) + (n-k)\log\Big(1-\frac{1}{r^k}\Big) - (n-k-1)\log\Big(1-\frac{2}{r^k}-\frac{1}{r^{k+1}}\Big)\Big) \\
&= -\log\Big(1-\frac{2}{r}\Big) + (n-1)\log\Big(1-\frac{1}{r}\Big) - (n-2)\log\Big(1-\frac{2}{r}-\frac{1}{r^2}\Big) - \\
&\quad - \log\Big(1-\frac{2}{r^2}\Big) + (n-2)\log\Big(1-\frac{1}{r^2}\Big) - (n-3)\log\Big(1-\frac{2}{r^2}-\frac{1}{r^3}\Big) + \\
&\quad + \sum_{k=3}^{n-1} \Big\{ -\log\Big(1-\frac{2}{r^k}\Big) + (n-k)\log\Big(1-\frac{1}{r^k}\Big) - (n-k-1)\log\Big(1-\frac{2}{r^k}-\frac{1}{r^{k+1}}\Big)\Big\}.
\end{aligned}$$

Now, employ the majorations valuable for $0 \leqslant \delta \leqslant 0.5$:

$$\begin{aligned}
\log\big(1-\delta\big) &\;\leqslant\; -\delta - \tfrac{1}{2}\,\delta^2, \\
-\log\big(1-\varepsilon\big) &\;\leqslant\; \varepsilon + \varepsilon^2,
\end{aligned}$$

to get using the assumption $r \geqslant 10$:

$$\begin{aligned}
\log \widehat{C}\big(\tfrac{1}{r}\big) \;\leqslant\; & \frac{2}{r}+\frac{4}{r^2}+(n-1)\left[-\frac{1}{r}-\frac{1}{2}\,\frac{1}{r^2}\right]+(n-2)\left[\Big(\frac{2}{r}+\frac{1}{r^2}\Big)+\underbrace{\Big(\frac{2}{r}+\frac{1}{r^2}\Big)^2}_{\leqslant \frac{5}{r^2}}\right]+\\
& +\underbrace{\frac{2}{r^2}+\frac{4}{r^4}}_{\leqslant \frac{3}{r^2}}+(n-2)\left[-\frac{1}{r^2}\underbrace{-\frac{1}{2}\,\frac{1}{r^4}}_{<0}\right]+(n-3)\left[\underbrace{\Big(\frac{2}{r^2}+\frac{1}{r^3}\Big)+\Big(\frac{2}{r^2}+\frac{1}{r^3}\Big)^2}_{\leqslant \frac{4}{r^2}}\right]+\\
& +\sum_{k=3}^{n-1}\left\{\underbrace{\frac{2}{r^k}+\frac{4}{r^{2k}}}_{\leqslant \frac{3}{r^k}}+\underbrace{(n-k)\log\Big(1-\frac{1}{r^k}\Big)}_{<0}+(n-k-1)\left[\underbrace{\Big(\frac{2}{r^k}+\frac{1}{r^{k+1}}\Big)+\Big(\frac{2}{r^k}+\frac{1}{r^{k+1}}\Big)^2}_{\leqslant \frac{3}{r^k}}\right]\right\}\\
\leqslant\; & \frac{1}{r}\big[2-n+1+2\,n-4\big]+\frac{1}{r^2}\big[4-\tfrac{n}{2}+\tfrac{1}{2}+n-2+5\,n-10+3-n+2+0+4\,n-12\big]\\
& +\sum_{k=3}^{n-1}\frac{3}{r^k}\big[1+0+n-k-1\big]\\
\leqslant\; & \frac{1}{r}\big[n-1\big]+\frac{1}{r^2}\big[\tfrac{17}{2}\,n-\tfrac{29}{2}\big]+\\
& +\frac{3}{r^3}\,\frac{1}{1-\frac{1}{r}}\big[n\big]\\
\leqslant\; & \frac{n}{r}+9\,\frac{n}{r^2}+3\,\frac{n}{r^2}. \qquad \square
\end{aligned}$$

Similarly to the expression:

$$\widehat{C}(x) \;=\; \prod_{k=1}^{n-1}\frac{1}{1-2\,x^k}\;\prod_{k=1}^{n-1}\big(1-x^k\big)^{n-k}\;\prod_{k=1}^{n-1}\frac{1}{(1-2\,x^k-x^{k+1})^{n-k-1}},$$

we obtain by simply changing the last sign $-$ to the sign $+$ in the denominator of the third product:

$$C(x) \;=\; \prod_{k=1}^{n-1}\frac{1}{1-2\,x^k}\;\prod_{k=1}^{n-1}\big(1-x^k\big)^{n-k}\;\prod_{k=1}^{n-1}\frac{1}{(1-2\,x^k+x^{k+1})^{n-k-1}}.$$

Lemma 9.2. *One has:*

$$\frac{\widehat{C}\big(\frac{1}{r}\big)}{C\big(\frac{1}{r}\big)} \;\leqslant\; e^{\,17\,\frac{n}{r^2}}.$$

Proof. This quotient writes as:

$$\frac{\widehat{C}\big(\frac{1}{r}\big)}{C\big(\frac{1}{r}\big)} \;=\; \prod_{k=1}^{n-2}\left(\frac{1-2\,\frac{1}{r^k}+\frac{1}{r^{k+1}}}{1-2\,\frac{1}{r^k}-\frac{1}{r^{k+1}}}\right)^{n-k-1},$$

since the terms for $k = n-1$ drop. Take logarithm and use the above majorations:

$$\begin{aligned}
\log \frac{\widehat{C}\left(\frac{1}{r}\right)}{C\left(\frac{1}{r}\right)} &= \sum_{k=1}^{n-2} (n-k-1) \left[\log\left(1 - \left(\frac{2}{r^k} - \frac{1}{r^{k+1}}\right)\right) - \log\left(1 - \left(\frac{2}{r^k} + \frac{1}{r^{k+1}}\right)\right)\right] \\
&= (n-2) \left[\log\left(1 - \left(\frac{2}{r} - \frac{1}{r^2}\right)\right) - \log\left(1 - \left(\frac{2}{r} + \frac{1}{r^2}\right)\right)\right] + \\
&\quad + (n-3) \left[\log\left(1 - \left(\frac{2}{r^2} - \frac{1}{r^3}\right)\right) - \log\left(1 - \left(\frac{2}{r^2} + \frac{1}{r^3}\right)\right)\right] + \\
&\quad + \sum_{k=3}^{n-2} (n-k-1) \left[\underbrace{\log\left(1 - \left(\frac{2}{r^k} - \frac{1}{r^{k+1}}\right)\right)}_{<0} - \log\left(1 - \left(\frac{2}{r^k} + \frac{1}{r^{k+1}}\right)\right)\right] \\
&\leqslant (n-2) \left[\underline{-\left(\frac{2}{r}\right.} - \frac{1}{r^2}) - \underbrace{\frac{1}{2}\left(\frac{2}{r} - \frac{1}{r^2}\right)^2}_{<0} + \underline{\left(\frac{2}{r}\right.} + \frac{1}{r^2}\Big) + \underbrace{\left(\frac{2}{r} + \frac{1}{r^2}\right)^2}_{\leqslant \frac{5}{r^2}}\right] + \\
&\quad + (n-3) \left[\underbrace{-\left(\frac{2}{r^2} - \frac{1}{r^3}\right)}_{<0} \underbrace{-\frac{1}{2}\left(\frac{2}{r^2} - \frac{1}{r^3}\right)^2}_{<0} + \underbrace{\left(\frac{2}{r^2} + \frac{1}{r^3}\right) + \left(\frac{2}{r^2} + \frac{1}{r^3}\right)^2}_{\leqslant \frac{4}{r^2}}\right] + \\
&\quad + \sum_{k=3}^{n-2} (n-k-1) \left[0 + 2 \Big(\underbrace{\frac{2}{r^k} + \frac{1}{r^{k+1}}}_{\leqslant \frac{3}{r^k}} \Big)\right], \qquad [\varepsilon + \varepsilon^2 \leqslant 2\,\varepsilon]
\end{aligned}$$

and notice, *importantly*, that the $\frac{1}{r}$-terms disappear, so that at the end:

$$\begin{aligned}
\log \frac{\widehat{C}\left(\frac{1}{r}\right)}{C\left(\frac{1}{r}\right)} &\leqslant (n-2) \left[\frac{7}{r^2}\right] + (n-3) \left[\frac{4}{r^2}\right] + 6\,n \sum_{k=3}^{\infty} \frac{1}{r^k} \\
&\leqslant 11\,n\,\frac{1}{r^2} + 6\,n\,\frac{1}{r^2}\,\frac{1}{r-1}. \qquad \square
\end{aligned}$$

Lastly, making the choice:

$$r := \sqrt{n}\,a(n),$$

with a function $a(n) \underset{n\to\infty}{\longrightarrow} \infty$ tending *slowly* to infinity — think $a(n) := \log\log n$ — and satisfying at least $a(n) \ll n^{\epsilon}$ for any $\epsilon > 0$, we want to minorize:

$$\boldsymbol{CR}_{\infty} := C\left(\tfrac{1}{r}\right) = C\left(\tfrac{1}{\sqrt{n}\,a(n)}\right).$$

Lemma 9.3. *One has:*

$$C\left(\tfrac{1}{\sqrt{n}\,a(n)}\right) \geqslant e^{\frac{1}{2}\frac{\sqrt{n}}{a(n)}} \underset{n\to\infty}{\longrightarrow} \infty.$$

Proof. Take logarithm:

$$\begin{aligned}
\log C\left(\tfrac{1}{r}\right) &= \sum_{k=1}^{n-1}\Big(-\log\Big(1-\frac{2}{r^k}\Big)+(n-k)\log\Big(1-\frac{1}{r^k}\Big)-(n-k-1)\log\Big(1-\Big(\frac{2}{r^k}-\frac{1}{r^{k+1}}\Big)\Big)\Big)\\
&= -\log\Big(1-\frac{2}{r}\Big)+(n-1)\log\Big(1-\frac{1}{r}\Big)-(n-2)\log\Big(1-\Big(\frac{2}{r}-\frac{1}{r^2}\Big)\Big)-\\
&\quad-\log\Big(1-\frac{2}{r^2}\Big)+(n-2)\log\Big(1-\frac{1}{r^2}\Big)-(n-3)\log\Big(1-\Big(\frac{2}{r^2}-\frac{1}{r^3}\Big)\Big)+\\
&\quad+\sum_{k=3}^{n-1}\Big\{-\log\Big(1-\frac{2}{r^k}\Big)+(n-k)\log\Big(1-\frac{1}{r^k}\Big)-(n-1-k)\log\Big(1-\Big(\frac{2}{r^k}-\frac{1}{r^{k+1}}\Big)\Big)\Big\},
\end{aligned}$$

use the minorations:

$$\begin{aligned}
-\log\big(1-\varepsilon\big) &\geqslant \varepsilon,\\
\log\big(1-\delta\big) &\geqslant -\delta-\delta^2,
\end{aligned}$$

to get:

$$\begin{aligned}
\log C\big(\tfrac{1}{r}\big) &\geqslant \frac{2}{r}+(n-1)\Big[-\frac{1}{r}-\frac{1}{r^2}\Big]+(n-2)\Big[\frac{2}{r}-\frac{1}{r^2}\Big]+\\
&\quad+\frac{2}{r^2}+(n-2)\Big[\underbrace{-\frac{1}{r^2}-\frac{1}{r^4}}_{\geqslant -\frac{2}{r^2}}\Big]+(n-3)\Big[\underbrace{\frac{2}{r^2}-\frac{1}{r^3}}_{\geqslant \frac{1}{r^2}}\Big]+\\
&\quad+\sum_{k=3}^{n-1}\Big\{\frac{2}{r^k}+(n-k)\Big[\underbrace{-\frac{1}{r^k}-\frac{1}{r^{2k}}}_{\geqslant -\frac{2}{r^k}}\Big]+(n-k-1)\Big[\underbrace{\frac{2}{r^k}-\frac{1}{r^{k+1}}}_{\geqslant \frac{1}{r^k}}\Big]\Big\}\\
&\geqslant \frac{1}{r}\big[2-n+1+2\,n-4\big]+\frac{1}{r^2}\big[-n+1-n+2+2-2\,n+4+n-3\big]+\\
&\quad+\sum_{k=3}^{n-1}\frac{1}{r^k}\big[2-2\,n+2\,k+n-k-1\big]\\
&= \frac{1}{r}\big[n-1\big]+\frac{1}{r^2}\big[-3\,n+6\big]+\sum_{k=3}^{n-1}\frac{1}{r^k}\big[\underbrace{-n+k+1}_{\geqslant -n+1}\big]\\
&\geqslant (n-1)\Big[\frac{1}{r}-\frac{3}{r^2}-\sum_{k=3}^{\infty}\frac{1}{r^k}\Big]\\
&\geqslant (n-1)\Big[\frac{1}{r}-\frac{4}{r^2}\Big].
\end{aligned}$$

Lastly, again with $a(n)\ll n^{\epsilon}$:

$$\begin{aligned}
\log C\Big(\frac{1}{\sqrt{n}\,a(n)}\Big) &\geqslant \frac{n-1}{\sqrt{n}\,a(n)}\underbrace{-\frac{4n-4}{n\,a(n)^2}}_{\geqslant -4}\\
&\geqslant \frac{1}{2}\frac{\sqrt{n}}{a(n)}.
\end{aligned}$$

□

10. Final minorations

As explained at the end of Section **4**, with a suitable choice of r, the goal is to show:

$$1 \overset{?}{\leqslant} \sum_{\substack{0\leqslant k_2\leqslant n\\ 0\leqslant k_3\leqslant n+k_2\\ \cdots\cdots\cdots\cdots\cdots\\ 0\leqslant k_{n-1}\leqslant n+k_{n-2}\\ 0\leqslant k_n\ \ \leqslant n+k_{n-1}}} C_{k_2,k_3,\ldots,k_{n-1},k_n}\, M_{k_2,k_3,\ldots,k_{n-1},k_n}\, \frac{1}{r^{k_2+k_3+\cdots+k_{n-1}+k_n}}$$

$$=: \boldsymbol{CMR}.$$

Abbreviate this domain range as:

$$\measuredangle\square := \Big\{\big(k_2,k_3,\ldots,k_{n-1},k_n\big)\in\mathbb{N}^n\colon\ \begin{aligned} &0\ \leqslant\ k_2\ \leqslant\ n,\\ &0\ \leqslant\ k_3\ \leqslant\ n+k_2,\\ &\cdots\cdots\cdots\cdots\cdots\\ &0\ \leqslant\ k_{n-1}\ \leqslant\ n+k_{n-2},\\ &0\ \leqslant\ k_n\ \ \leqslant\ n+k_{n-1}\Big\}. \end{aligned}$$

Observe that:

$$\Big\{k_2+k_3+\cdots+k_{n-1}+k_n\ \leqslant\ n\Big\}\ \subset\ \measuredangle\square,$$

hence *a fortiori* with a function $c(n)\underset{n\to\infty}{\longrightarrow}\infty$ tending slowly to infinity to be chosen later:

$$\Big\{k_2+k_3+\cdots+k_{n-1}+k_n\ \leqslant\ \tfrac{\sqrt{n}}{c(n)}\Big\}\ \subset\ \measuredangle\square.$$

Introduce:

$$\boldsymbol{CMR}_{\mathrm{T}} := \sum_{k_2+\cdots+k_n\leqslant\frac{\sqrt{n}}{c(n)}} C_{k_2,\ldots,k_n}\, M_{k_2,\ldots,k_n}\, \frac{1}{r^{k_2+\cdots+k_n}},$$

the letter 'T' standing for 'Truncated', with the Remainder:

$$\begin{aligned}\boldsymbol{CMR}-\boldsymbol{CMR}_{\mathrm{T}} &= \sum_{\substack{(k_2,\ldots,k_n)\in\measuredangle\square\\ k_2+\cdots+k_n\geqslant 1+\frac{\sqrt{n}}{c(n)}}} C_{k_2,\ldots,k_n}\, M_{k_2,\ldots,k_n}\, \frac{1}{r^{k_2+\cdots+k_n}}\\ &=: \boldsymbol{CMR}_{\mathrm{R}}.\end{aligned}$$

Along with these quantities, introduce also:

$$\boldsymbol{CMR}^{+}_{\mathrm{T}} := \sum_{\substack{k_2+\cdots+k_n\leqslant\frac{\sqrt{n}}{c(n)}\\ C_{k_2,\ldots,k_n}\geqslant 0}} C_{k_2,\ldots,k_n}\, M_{k_2,\ldots,k_n}\, \frac{1}{r^{k_2+\cdots+k_n}} \qquad (\geqslant 0),$$

$$\boldsymbol{CMR}^{-}_{\mathrm{T}} := \sum_{\substack{k_2+\cdots+k_n\leqslant\frac{\sqrt{n}}{c(n)}\\ C_{k_2,\ldots,k_n}<0}} \big(-C_{k_2,\ldots,k_n}\big)\, M_{k_2,\ldots,k_n}\, \frac{1}{r^{k_2+\cdots+k_n}} \qquad (\geqslant 0),$$

two nonnegative quantities which decompose:

$$\boldsymbol{CMR}_{\mathrm{T}}\ =\ \boldsymbol{CMR}^{+}_{\mathrm{T}}-\boldsymbol{CMR}^{-}_{\mathrm{T}}.$$

In addition, without the multinomial-quotient coefficients, introduce:

$$\begin{aligned}
\boldsymbol{CR} &:= \sum_{(k_2,\dots,k_n)\in\measuredangle\square} C_{k_2,\dots,k_n}\cdot 1\cdot\frac{1}{r^{k_2+\cdots+k_n}},\\
\boldsymbol{CR}_{\mathrm{T}} &:= \sum_{k_2+\cdots+k_n\leqslant\frac{\sqrt{n}}{c(n)}} C_{k_2,\dots,k_n}\cdot 1\cdot\frac{1}{r^{k_2+\cdots+k_n}},\\
\boldsymbol{CR}_{\mathrm{R}} &:= \sum_{\substack{(k_2,\dots,k_n)\in\measuredangle\square\\ k_2+\cdots+k_n\geqslant 1+\frac{\sqrt{n}}{c(n)}}} C_{k_2,\dots,k_n}\cdot 1\cdot\frac{1}{r^{k_2+\cdots+k_n}},
\end{aligned}$$

and similarly also:

$$\begin{aligned}
\boldsymbol{CR}^{+}_{\mathrm{T}} &:= \sum_{\substack{k_2+\cdots+k_n\leqslant\frac{\sqrt{n}}{c(n)}\\ C_{k_2,\dots,k_n}\geqslant 0}} C_{k_2,\dots,k_n}\cdot 1\cdot\frac{1}{r^{k_2+\cdots+k_n}} \qquad (\geqslant 0),\\
\boldsymbol{CR}^{-}_{\mathrm{T}} &:= \sum_{\substack{k_2+\cdots+k_n\leqslant\frac{\sqrt{n}}{c(n)}\\ C_{k_2,\dots,k_n}< 0}} \big(-C_{k_2,\dots,k_n}\big)\cdot 1\cdot\frac{1}{r^{k_2+\cdots+k_n}} \qquad (\geqslant 0).
\end{aligned}$$

Recall that we are choosing:

$$r(n) \,=\, \sqrt{n}\,a(n),$$

and we now endeavor to find a condition guaranteeing that the remainder $\big|\boldsymbol{CMR}_{\mathrm{R}}\big|$ be small in absolute value.

To this aim, choose in the Cauchy inequalities $\rho := \frac{1}{\sqrt{n}}$, apply Lemma 9.1:

$$\widehat{C}\big(\tfrac{1}{\sqrt{n}}\big) \;\leqslant\; e^{12}\,e^{\sqrt{n}},$$

so that Observation 8.2 gives:

$$\widehat{C}_h \;\leqslant\; \frac{1}{\big(\frac{1}{\sqrt{n}}\big)^h}\,e^{12}\,e^{\sqrt{n}} \qquad (\forall\, h\geqslant 0).$$

Now, majorize the remainder:

$$\begin{aligned}
\big|\boldsymbol{CMR}_{\mathrm{R}}\big| &= \Bigg| \sum_{\substack{(k_2,\dots,k_n)\in\measuredangle\square \\ k_2+\cdots+k_n\geqslant 1+\frac{\sqrt{n}}{c(n)}}} C_{k_2,\dots,k_n}\, M_{k_2,\dots,k_n}\, \frac{1}{r^{k_2+\cdots+k_n}} \Bigg| \\
&\leqslant \sum_{\substack{(k_2,\dots,k_n)\in\measuredangle\square \\ k_2+\cdots+k_n\geqslant 1+\frac{\sqrt{n}}{c(n)}}} \big|C_{k_2,\dots,k_n}\big| \cdot 1 \cdot \frac{1}{r^{k_2+\cdots+k_n}} \\
&\leqslant \sum_{k_2+\cdots+k_n\geqslant 1+\frac{\sqrt{n}}{c(n)}} \big|C_{k_2,\dots,k_n}\big| \, \frac{1}{r^{k_2+\cdots+k_n}} \\
&\leqslant \sum_{k_2+\cdots+k_n\geqslant 1+\frac{\sqrt{n}}{c(n)}} \widehat{C}_{k_2,\dots,k_n} \, \frac{1}{r^{k_2+\cdots+k_n}} \\
&= \sum_{h=1+\frac{\sqrt{n}}{c(n)}}^{\infty} \frac{1}{r^h} \sum_{k_2+\cdots+k_n=h} \widehat{C}_{k_2,\dots,k_n} \\
&= \sum_{h=1+\frac{\sqrt{n}}{c(n)}}^{\infty} \frac{1}{r^h}\, \widehat{C}_h,
\end{aligned}$$

and hence, thanks to what precedes:

$$\begin{aligned}
\big|\boldsymbol{CMR}_{\mathrm{R}}\big| &\leqslant \sum_{h=1+\frac{\sqrt{n}}{c(n)}}^{\infty} \frac{1}{r^h} \frac{1}{\big(\frac{1}{\sqrt{n}}\big)^h}\, e^{12}\, e^{\sqrt{n}} \\
&= e^{12}\, e^{\sqrt{n}} \sum_{h=1+\frac{\sqrt{n}}{c(n)}}^{\infty} \frac{1}{\Big(\underline{\sqrt{n}}_{\circ}\, a(n)\, \frac{1}{\underline{\sqrt{n}}_{\circ}}\Big)^h} \\
&= e^{12}\, e^{\sqrt{n}}\, \frac{1}{\big(a(n)\big)^{1+\frac{\sqrt{n}}{c(n)}}} \sum_{h=0}^{\infty} \Big(\frac{1}{a(n)}\Big)^h \\
&= e^{12}\, e^{\sqrt{n}}\, e^{-\left(1+\frac{\sqrt{n}}{c(n)}\right)\log a(n)} \underbrace{\frac{1}{1-\frac{1}{a(n)}}}_{\leqslant 2} \\
&\leqslant 2\, e^{12}\, e^{-\log a(n)}\, e^{\sqrt{n}\left[1-\frac{\log a(n)}{c(n)}\right]}.
\end{aligned}$$

In order to insure that the right-hand side is small, since $e^{-\log a(n)} \underset{n\to\infty}{\longrightarrow} 0$, it suffices to choose:

$$c(n) := \log a(n) \underset{n\to\infty}{\longrightarrow} \infty,$$

to obtain:

$$\big|\boldsymbol{CMR}_{\mathrm{R}}\big| \leqslant 2\, e^{12}\, e^{-\log a(n)} \underset{n\to\infty}{\longrightarrow} 0.$$

Lemma 10.1. *With $c(n) = \log a(n)$, it holds:*

$$\begin{aligned}\big|\boldsymbol{CMR}_{\mathrm{R}}\big| \;&\leqslant \sum_{k_2+\cdots+k_n \geqslant 1+\frac{\sqrt{n}}{c(n)}} \widehat{C}_{k_2,\ldots,k_n}\,\frac{1}{r^{k_2+\cdots+k_n}}\\ &\leqslant\; 2\,e^{12}\,e^{-\log a(n)}.\end{aligned}$$

□

Further, it is now necessary to estimate the size of the first terms $\boldsymbol{CMR}_{\mathrm{T}}$, and to show that they are large. It will be useful that:

$$\frac{\sqrt{n}}{\log a(n)} \;=\; \mathrm{o}\big(\sqrt{n}\big) \qquad (n\longrightarrow\infty).$$

Introduce the quantities:

$$\begin{aligned}\boldsymbol{CR}_{\infty} \;&:=\; \sum_{k_2,\ldots,k_n\geqslant 0} C_{k_2,\ldots,k_n}\,\frac{1}{r^{k_2+\cdots+k_n}} \;=\; C^{n-1}\big(\tfrac{1}{r}\big),\\ \boldsymbol{CR}^{+}_{\infty} \;&:=\; \sum_{\substack{k_2,\ldots,k_n\geqslant 0\\ C_{k_2,\ldots,k_n}\geqslant 0}} C_{k_2,\ldots,k_n}\,\frac{1}{r^{k_2+\cdots+k_n}},\\ \boldsymbol{CR}^{-}_{\infty} \;&:=\; \sum_{\substack{k_2,\ldots,k_n\geqslant 0\\ C_{k_2,\ldots,k_n}< 0}} \big(-C_{k_2,\ldots,k_n}\big)\,\frac{1}{r^{k_2+\cdots+k_n}},\\ \widehat{\boldsymbol{CR}}_{\infty} \;&:=\; \sum_{k_2,\ldots,k_n\geqslant 0} \widehat{C}_{k_2,\ldots,k_n}\,\frac{1}{r^{k_2+\cdots+k_n}} \;=\; \widehat{C}^{n-1}\big(\tfrac{1}{r}\big),\end{aligned}$$

for which it is clear that:

$$\boldsymbol{CR}^{+}_{\infty}+\boldsymbol{CR}^{-}_{\infty} \;\leqslant\; \widehat{\boldsymbol{CR}}_{\infty}.$$

By Lemma 9.2:

$$\frac{\widehat{\boldsymbol{CR}}_{\infty}}{\boldsymbol{CR}_{\infty}} \;\leqslant\; e^{\frac{17}{a(n)^2}},$$

and next:

$$\boldsymbol{CR}^{+}_{\infty}+\boldsymbol{CR}^{-}_{\infty} \;\leqslant\; \widehat{\boldsymbol{CR}}_{\infty} \;\leqslant\; e^{\frac{17}{a(n)^2}}\,\boldsymbol{CR}_{\infty} \;=\; e^{\frac{17}{a(n)^2}}\,\Big(\boldsymbol{CR}^{+}_{\infty}-\boldsymbol{CR}^{-}_{\infty}\Big),$$

from which it comes:

$$\boldsymbol{CR}^{-}_{\infty}\,\Big(\underbrace{1+e^{\frac{17}{a(n)^2}}}_{\geqslant 2}\Big) \;\leqslant\; \Big(e^{\frac{17}{a(n)^2}}-1\Big)\,\boldsymbol{CR}^{+}_{\infty},$$

whence:

$$\boldsymbol{CR}^{-}_{\infty} \;\leqslant\; \tfrac{1}{2}\,\Big(e^{\frac{17}{a(n)^2}}-1\Big)\,\boldsymbol{CR}^{+}_{\infty}. \tag{10.2}$$

Next, we want to minorize $\boldsymbol{CMR}_{\mathrm{T}}$ in order to show it is large:

$$\begin{aligned}
\boldsymbol{CMR}_{\mathrm{T}} &= \boldsymbol{CMR}^{+}_{\mathrm{T}} - \boldsymbol{CMR}^{-}_{\mathrm{T}} \\
&= \sum_{\substack{k_2+\cdots+k_n \leqslant \frac{\sqrt{n}}{c(n)} \\ C_{k_2,\ldots,k_n} \geqslant 0}} C_{k_2,\ldots,k_n}\, M_{k_2,\ldots,k_n}\, \frac{1}{r^{k_2+\cdots+k_n}} - \\
&\quad - \sum_{\substack{k_2+\cdots+k_n \leqslant \frac{\sqrt{n}}{c(n)} \\ C_{k_2,\ldots,k_n} < 0}} \big(-\, C_{k_2,\ldots,k_n}\big)\, M_{k_2,\ldots,k_n}\, \frac{1}{r^{k_2+\cdots+k_n}} \\
&\qquad\qquad \text{[Proposition 5.4]} \\
&\geqslant \sum_{\substack{k_2+\cdots+k_n \leqslant \frac{\sqrt{n}}{c(n)} \\ C_{k_2,\ldots,k_n} \geqslant 0}} C_{k_2,\ldots,k_n}\, e^{-\frac{2}{c(n)^2}}\, \frac{1}{r^{k_2+\cdots+k_n}} - \\
&\qquad\qquad \text{[Lemma 5.1]} \\
&\quad - \sum_{\substack{k_2+\cdots+k_n \leqslant \frac{\sqrt{n}}{c(n)} \\ C_{k_2,\ldots,k_n} < 0}} \big(-\, C_{k_2,\ldots,k_n}\big) \cdot 1 \cdot \frac{1}{r^{k_2+\cdots+k_n}} \\
&= e^{-\frac{2}{c(n)^2}}\, \boldsymbol{CR}^{+}_{\mathrm{T}} - \boldsymbol{CR}^{-}_{\mathrm{T}},
\end{aligned}$$

but we yet need to compare these to the quantities $\boldsymbol{CR}^{\pm}_{\infty}$. Hence we estimate:

$$\begin{aligned}
\big|\boldsymbol{CR}^{+}_{\infty} - \boldsymbol{CR}^{+}_{\mathrm{T}}\big| \;=\; \boldsymbol{CR}^{+}_{\infty} - \boldsymbol{CR}^{+}_{\mathrm{T}} \;&=\; \sum_{\substack{k_2+\cdots+k_n \geqslant 1+\frac{\sqrt{n}}{c(n)} \\ C_{k_2,\ldots,k_n} \geqslant 0}} C_{k_2,\ldots,k_n}\, \frac{1}{r^{k_2+\cdots+k_n}} \\
&\leqslant \sum_{k_2+\cdots+k_n \geqslant 1+\frac{\sqrt{n}}{c(n)}} \widehat{C}_{k_2,\ldots,k_n}\, \frac{1}{r^{k_2+\cdots+k_n}} \\
&\leqslant\; 2\, e^{12}\, e^{-\log a(n)}
\end{aligned}$$

and more simply:

$$-\,\boldsymbol{CR}^{-}_{\mathrm{T}} \;\geqslant\; -\,\boldsymbol{CR}^{-}_{\infty},$$

since:

$$0 \;\leqslant\; \boldsymbol{CR}^{-}_{\infty} - \boldsymbol{CR}^{-}_{\mathrm{T}} \;=\; \sum_{\substack{k_2+\cdots+k_n \geqslant 1+\frac{\sqrt{n}}{c(n)} \\ C_{k_2,\ldots,k_n} < 0}} \big(-\, C_{k_2,\ldots,k_n}\big)\, \frac{1}{r^{k_2+\cdots+k_n}}.$$

Thanks to all this:

$$\begin{aligned}
\boldsymbol{CMR}_{\mathrm{T}} &\geqslant e^{-\frac{2}{c(n)^2}}\, \boldsymbol{CR}^{+}_{\mathrm{T}} - \boldsymbol{CR}^{-}_{\mathrm{T}} \\
&\geqslant e^{-\frac{2}{c(n)^2}} \Big[\boldsymbol{CR}^{+}_{\infty} - 2\, e^{12}\, e^{-\log a(n)}\Big] - \boldsymbol{CR}^{-}_{\infty}
\end{aligned}$$

hence applying the minoration (10.2) for $-\,\boldsymbol{CR}^{-}_{\infty}$:

$$\begin{aligned}
\boldsymbol{CMR}_{\mathrm{T}} &\geqslant e^{-\frac{2}{c(n)^2}} \Big[\boldsymbol{CR}^{+}_{\infty} - 2\, e^{12}\, e^{-\log a(n)}\Big] - \tfrac{1}{2} \Big(e^{\frac{17}{a(n)^2}} - 1\Big)\, \boldsymbol{CR}^{+}_{\infty} \\
&= \boldsymbol{CR}^{+}_{\infty} \Big[\underbrace{e^{-\frac{2}{(\log a(n))^2}}}_{\underset{n\to\infty}{\longrightarrow} 1} - \tfrac{1}{2} \big(\underbrace{e^{\frac{17}{a(n)^2}} - 1}_{\underset{n\to\infty}{\longrightarrow} 0}\big)\Big] - \underbrace{2\, e^{12}\, e^{-\log a(n)}\, e^{-\frac{2}{(\log a(n))^2}}}_{\underset{n\to\infty}{\longrightarrow} 0}.
\end{aligned}$$

Since trivially:

$$\begin{aligned} \boldsymbol{CR}_{\infty}^{+} &= \boldsymbol{CR}_{\infty} + \boldsymbol{CR}_{\infty}^{-} \\ &\geqslant \boldsymbol{CR}_{\infty}, \end{aligned}$$

it comes:

$$\boldsymbol{CMR}_{\mathrm{T}} \geqslant \boldsymbol{CR}_{\infty} \left[e^{-\frac{2}{(\log a(n))^2}} - \tfrac{1}{2} \left(e^{\frac{17}{a(n)^2}} - 1 \right) \right] - 2\, e^{12}\, e^{-\log a(n)}\, e^{-\frac{2}{(\log a(n))^2}},$$

whence using Lemma 9.3:

$$\boldsymbol{CMR}_{\mathrm{T}} \geqslant e^{\frac{1}{2}\frac{\sqrt{n}}{a(n)}} \left[e^{-\frac{2}{(\log a(n))^2}} - \tfrac{1}{2} \left(e^{\frac{17}{a(n)^2}} - 1 \right) \right] - 2\, e^{12}\, e^{-\log a(n)}\, e^{-\frac{2}{(\log a(n))^2}}.$$

Coming back to:

$$\begin{aligned} \boldsymbol{CMR} &\geqslant \boldsymbol{CMR}_{\mathrm{T}} - \left| \boldsymbol{CMR}_{\mathrm{R}} \right| \\ &\geqslant \boldsymbol{CMR}_{\mathrm{T}} - 2\, e^{12}\, e^{-\log a(n)}\, e^{-\frac{2}{(\log a(n))^2}}, \end{aligned}$$

we obtain finally:

$$\boldsymbol{CMR} \geqslant e^{\frac{1}{2}\frac{\sqrt{n}}{a(n)}} \left[e^{-\frac{2}{(\log a(n))^2}} - \tfrac{1}{2} \left(e^{\frac{17}{a(n)^2}} - 1 \right) \right] - 2\, e^{12}\, e^{-\log a(n)} \left(1 + e^{-\frac{2}{(\log a(n))^2}} \right).$$

This minorant is $\geqslant 1$ for all $n \gg \mathrm{N}$ large enough.

Proof of Theorem 1.1. Choose — think integer value —:

$$r := \sqrt{n}\, \frac{\log n}{2}.$$

Proposition 3.1 concludes for $\mathbb{X}^{n-1} \subset \mathbb{P}^n$ of degree at least:

$$\begin{aligned} d_{\mathrm{GG}}(n) &:= \left(\sqrt{n}\, \frac{\log n}{2} + 3 \right)^n 25\, n^2 \\ &= \left(\sqrt{n} \log n \right)^n \frac{1}{2^n} \left(1 + \frac{6}{\sqrt{n} \log n} \right)^n 25\, n^2 \\ &= \left(\sqrt{n} \log n \right)^n e^{-n \log 2}\, e^{n \log \left(1 + \frac{6}{\sqrt{n} \log n} \right)} 25\, n^2 \\ &\leqslant \left(\sqrt{n} \log n \right)^n \underbrace{e^{-n \log 2 + \frac{6\sqrt{n}}{\log n}}\, 25\, n^2}_{< 1 \text{ when } n \geqslant \mathrm{N}_{\mathrm{GG}}}, \end{aligned}$$

since it is visible that this majorant becomes < 1 when n is large enough, say for all $n \geqslant \mathrm{N}_{\mathrm{GG}}$, where N_{GG} can be made effective. □

Proof of Theorem 1.2. Thanks to [15], Proposition 3.1 concludes for $\mathbb{X}^{n-1} \subset \mathbb{P}^n$ of degree at least:

$$\begin{aligned}
d_{\mathrm{K}}(n) &:= \left(\sqrt{2\,n}\,\frac{\log\log(2\,n)}{2}+3\right)^{2n} 25\,(2\,n)^2 \\
&= \left(\frac{\sqrt{n}}{\sqrt{2}}\log\log(2\,n)\right)^{2n}\left(1+\frac{3\,\sqrt{2}}{\sqrt{n}\log\log(2\,n)}\right)^{2n} 100\,n^2 \\
&= \Big(n\log n\Big)^n\,\frac{\big(\log\log(2\,n)\big)^{2n}}{\big(\log n\big)^n}\,\frac{1}{\sqrt{2}^{\,2n}}\left(1+\frac{3\,\sqrt{2}}{\sqrt{n}\log\log(2\,n)}\right)^{2n} 100\,n^2 \\
&= \Big(n\log n\Big)^n\,e^{2n\log\log\log(2\,n)-n\log\log n-n\log 2+2n\log\left(1+\frac{3\,\sqrt{2}}{\sqrt{n}\log\log(2\,n)}\right)}\,100\,n^2 \\
&\leqslant \Big(n\log n\Big)^n\,\underbrace{e^{2n\log\log\log(2\,n)-n\log\log n-n\log 2+\frac{2\,\sqrt{n}\,3\,\sqrt{2}}{\log\log(2\,n)}}\,100\,n^2}_{<\,1\ \text{when}\ n\,\geqslant\,\mathrm{N}_{\mathrm{K}}}.
\end{aligned}$$

□

11. Some inequalities on circles $\{|z|=\rho\}$

Proposition 11.1. *For every radius $0 \leqslant \rho \leqslant \frac{1}{4}$, the functions:*

$$G_k(z) := \frac{1-z^k}{1-2\,z^k} \qquad (k\geqslant 1),$$

$$H_\ell(z) := \frac{1-z^\ell}{1-2\,z^\ell-z^{\ell+1}}, \qquad (\ell\geqslant 1)$$

attain, on the circle $\big\{z\in\mathbb{C}\colon |z|=\rho\big\}$, their maximum modulus at the real point $z=\rho$:

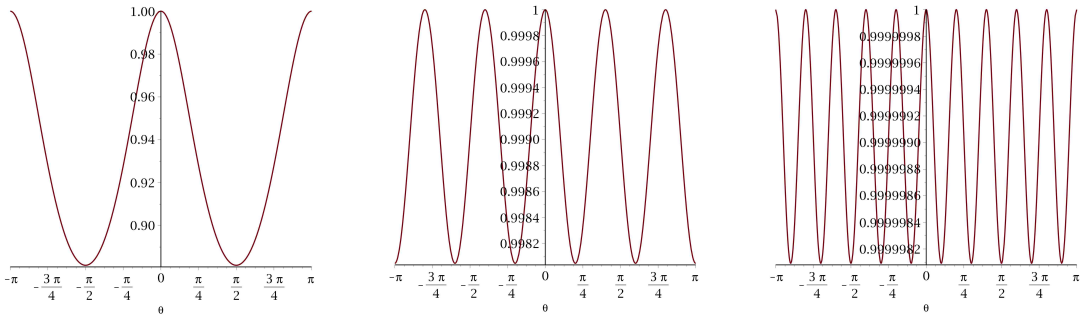
$$\max_{|z|=\rho}\left|\frac{1-z^k}{1-2\,z^k}\right| = \frac{1-\rho^k}{1-2\,\rho^k} \qquad (\forall\, k\geqslant 1),$$

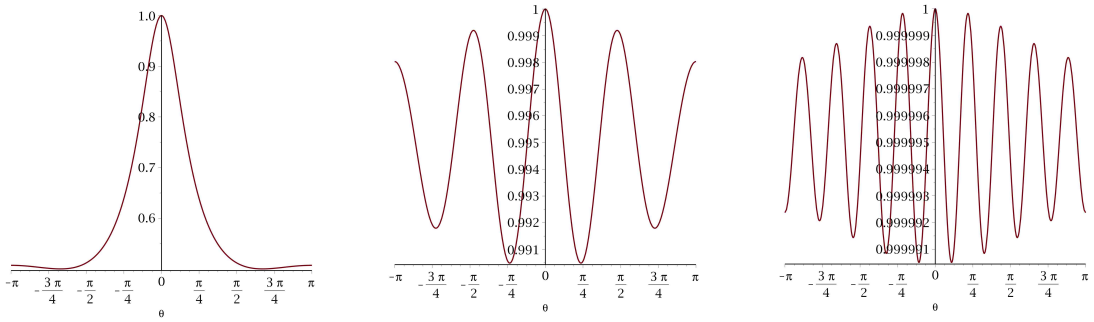
$$\max_{|z|=\rho}\left|\frac{1-z^\ell}{1-2\,z^\ell-z^{\ell+1}}\right| = \frac{1-\rho^\ell}{1-2\,\rho^\ell-\rho^{\ell+1}} \qquad (\forall\, \ell\geqslant 1),$$

and with the choice $\rho := 0.25$, the graphs on the unit circle of the two quotient functions:

$$\theta \longmapsto \frac{\big|G_k(\rho\,e^{i\theta})\big|}{G_k(\rho)} \qquad \textit{and} \qquad \theta \longmapsto \frac{\big|H_\ell(\rho\,e^{i\theta})\big|}{H_\ell(\rho)} \qquad (-\pi\leqslant\theta\leqslant\pi)$$

show up, respectively, for the three choices $k=2,5,10$ and the three choices $\ell=2,5,10$, as:





Proof. Treat at first the $G_k\big(\rho\, e^{i\theta}\big)$ with $\theta \in \mathbb{R}$, by squaring:

$$\frac{\big|1-\rho^k\, e^{ik\theta}\big|^2}{\big|1-2\,\rho^k\, e^{ik\theta}\big|^2} \overset{?}{\leqslant} \frac{\big(1-\rho^k\big)^2}{\big(1-2\,\rho^k\big)^2} \qquad (\forall\,\theta\in\mathbb{R}),$$

that is to say:

$$\frac{\big(1-\rho^k\cos k\theta\big)^2+\big(-\,\rho^k\sin k\theta\big)^2}{\big(1-2\,\rho^k\cos k\theta\big)^2+\big(-\,2\,\rho^k\sin k\theta\big)^2} \overset{?}{\leqslant} \frac{1-2\,\rho^k+\rho^{2k}}{1-4\,\rho^k+4\,\rho^{2k}},$$

or equivalently, after crossing/clearing the fractions:

$$\begin{aligned}
0 \overset{?}{\leqslant}\ & \big(1-2\,\rho^k+\rho^{2k}\big)\big(1-4\,\rho^k\cos k\theta+4\,\rho^{2k}\big)-\big(1-4\,\rho^k+4\,\rho^{2k}\big)\big(1-2\,\rho^k\cos k\theta+\rho^{2k}\big)\\
=\ & \underline{1}_{\circ}-4\,\rho^k\cos k\theta+\underline{4\,\rho^{2k}}_{\circ}\\
& \quad -\,2\,\rho^k\qquad\quad +\underline{8\,\rho^{2k}\cos k\theta}_{\circ}-8\,\rho^{3k}\\
& \qquad\qquad\quad +\underline{\rho^{2k}}_{\circ}\qquad\quad -\,4\,\rho^{3k}\cos k\theta+\underline{4\,\rho^{4k}}_{\circ}\\
& -\underline{1}_{\circ}+2\,\rho^k\cos k\theta-\underline{\rho^{2k}}_{\circ}\\
& \quad +4\,\rho^k\qquad\quad -\underline{8\,\rho^{2k}\cos k\theta}_{\circ}+4\,\rho^{3k}\\
& \qquad\qquad\quad -\,\underline{4\,\rho^{2k}}_{\circ}\qquad\quad +8\,\rho^{3k}\cos k\theta-\underline{4\,\rho^{4k}}_{\circ}.
\end{aligned}$$

Visibly, $5\cdot 2 = 10$ underlined terms annihilate by pairs:

$$\begin{aligned}
0 \overset{?}{\leqslant}\ & 2\,\rho^k-2\,\rho^k\cos k\theta-4\,\rho^{3k}+4\,\rho^{3k}\cos k\theta\\
=\ & 2\,\rho^k\Big[1-\cos k\theta-2\,\rho^{2k}+2\,\rho^{2k}\cos k\theta\Big],
\end{aligned}$$

and by luck, the obtained expression factorizes under a form which shows well that it takes only nonnegative values because $0\leqslant\rho\leqslant 0.25$:

$$0 \overset{\text{yes}}{\leqslant} 2\,\rho^k\big(1-2\,\rho^{2k}\big)\big(1-\cos k\theta\big) \qquad (\forall\,k\geqslant 1,\ \forall\,\theta\in\mathbb{R}).$$

Secondly, for the functions $\big(H_\ell(z)\big)_{\ell\geqslant 1}$, no such pleasant factorization is available. One can then view these $H_\ell(z)$ as 'perturbations' of the $G_\ell(z)$, with the addition of $-\,z^{\ell+1}$ at the denominator. More precisely, starting from the desired inequality of which we take the squared modulus:

$$\frac{\big|1-\rho^\ell\, e^{i\ell\theta}\big|^2}{\big|1-2\,\rho^\ell\, e^{i\ell\theta}-\underline{\rho^{\ell+1}\, e^{i(\ell+1)\,\theta}}\big|^2} \overset{?}{\leqslant} \frac{\big(1-\rho^\ell\big)^2}{\big(1-2\,\rho^\ell-\underline{\rho^{\ell+1}}\big)^2},$$

the 'perturbing terms' being underlined, after crossing/clearing the denominators, we are led to establish an inequality which is a 'perturbation' of the one just done above:

$$\begin{aligned}0 \overset{?}{\leqslant}\ & \big(1-2\,\rho^{\ell}+\rho^{2\ell}\big)\Big[\big(1-2\,\rho^{\ell}\cos\ell\theta-\underline{\rho^{\ell+1}\cos(\ell+1)\theta}\big)^2+\big(2\,\rho^{\ell}\sin\ell\theta+\underline{\rho^{\ell+1}\sin(\ell+1)\theta}\big)^2\Big]\\ -\ & \big(1-4\,\rho^{\ell}+4\,\rho^{2\ell}-\underline{2\,\rho^{\ell+1}+4\,\rho^{2\ell+1}+\rho^{2\ell+2}}\big)\Big[\big(1-\rho^{\ell}\cos\ell\theta\big)^2+\big(\rho^{\ell}\sin\ell\theta\big)^2\Big],\end{aligned}$$

the perturbing terms being still underlined, that is to say:

$$\begin{aligned}0 \overset{?}{\leqslant}\ & \big(1-2\,\rho^{\ell}+\rho^{2\ell}\big)\Big[1-4\,\rho^{\ell}\cos\ell\theta+4\,\rho^{2\ell}\cos^2\ell\theta+4\,\rho^{2\ell}\sin^2\ell\theta\\ & \qquad -\ \underline{2\,\rho^{\ell+1}\cos(\ell+1)\theta+4\,\rho^{2\ell+1}\cos\ell\theta\cos(\ell+1)\theta+\rho^{2\ell+2}\cos^2(\ell+1)\theta}\\ & \qquad\qquad \underline{+\,4\,\rho^{2\ell+1}\sin\ell\theta\sin(\ell+1)\theta+\rho^{2\ell+2}\sin^2(\ell+1)\theta}\Big]\\ -\ & \big(1-4\,\rho^{\ell}+4\,\rho^{2\ell}-\underline{2\,\rho^{\ell+1}+4\,\rho^{2\ell+1}+\rho^{2\ell+2}}\big)\Big[1-2\,\rho^{\ell}\cos\ell\theta+\rho^{2\ell}\cos^2\ell\theta+\rho^{2\ell}\sin^2\ell\theta\Big].\end{aligned}$$

Without redoing the calculation concerning the (principal, not underlined) terms, and using:

$$\cos(\ell+1)\theta\cos(-\ell\theta)-\sin(\ell+1)\theta\sin(-\ell\theta)\ =\ \cos\big((\ell+1-\ell)\,\theta\big),$$

we obtain:

$$\begin{aligned}0 \overset{?}{\leqslant}\ & 2\,\rho^{\ell}\big(1-2\,\rho^{2\ell}\big)\big(1-\cos\ell\theta\big)\\ & +\big(1-2\,\rho^{\ell}+\rho^{2\ell}\big)\Big[-2\,\rho^{\ell+1}\cos(\ell+1)\theta+4\,\rho^{2\ell+1}\cos\theta+\rho^{2\ell+2}\Big]\\ & +\big(2\,\rho^{\ell+1}-4\,\rho^{2\ell+1}-\rho^{2\ell+2}\big)\Big[1-2\,\rho^{\ell}\cos\ell\theta+\rho^{2\ell}\Big].\end{aligned}$$

Now, organize the expansion of lines 2 and 3 in a convenient synoptic way:

$$\begin{array}{llll}
-\,2\,\rho^{\ell+1}\cos(\ell+1)\theta+4\,\rho^{2\ell+1}\cos\theta & & +\,\underline{\rho^{2\ell+2}}_{\circ} & \\
 & +\,4\,\rho^{2\ell+1}\cos(\ell+1)\theta & -\,8\,\rho^{3\ell+1}\cos\theta-2\,\rho^{3\ell+2} & \\
 & & -\,2\,\rho^{3\ell+1}\cos(\ell+1)\theta+4\,\rho^{4\ell+1}\cos\theta+\underline{\rho^{4\ell+2}}_{\circ} & \\
+\,2\,\rho^{\ell+1} & -\,4\,\rho^{2\ell+1}\cos\ell\theta & +\,2\,\rho^{3\ell+1} & \\
 & -\,4\,\rho^{2\ell+1} & +\,8\,\rho^{3\ell+1}\cos\ell\theta & -\,4\,\rho^{4\ell+1} \\
 & & -\,\underline{\rho^{2\ell+2}}_{\circ} \qquad +\,2\,\rho^{3\ell+2}\cos\ell\theta & -\,\underline{\rho^{4\ell+2}}_{\circ}.
\end{array}$$

Only $4=2\cdot 2$ terms annihilate by pairs, and the question is reduced to determine whether there is nonnegativity:

$$\begin{aligned}0 \overset{?}{\leqslant}\ & 2\,\rho^{\ell}\big(1-2\,\rho^{2\ell}\big)\big(1-\cos\ell\theta\big)+2\,\rho^{\ell+1}\big(1-\cos(\ell+1)\theta\big)\\ & +\rho^{2\ell+1}\Big[4\cos\theta+4\cos(\ell+1)\theta-4\cos\ell\theta-4\Big]\\ & +\rho^{3\ell+1}\Big[-\,8\cos\theta-2\cos(\ell+1)\theta+8\cos\ell\theta+2\Big]\\ & +\rho^{3\ell+2}\Big[2\cos\ell\theta-2\Big]+\rho^{4\ell+1}\Big[4\cos\theta-4\Big]\\ =:\ & f_{\ell,\rho}(\theta),\end{aligned}$$

for a certain family $\big(f_{\ell,\rho}\big)_{1\leqslant\ell}^{0\leqslant\rho\leqslant 1/4}$ of 2π-periodic functions. Since this is trivially satisfied when $\rho = 0$, we shall from now on assume that:

$$0 \ < \ \rho \ \leqslant \ \tfrac{1}{4} \ = \ 0.25 \qquad \text{(assumption)}.$$

For the first term of $f_{\ell,\rho}$, since we have:

$$1 \ < \ 2\big(1 - 2\cdot 0.25^2\big) \ \leqslant \ 2\big(1 - 2\,\rho^{2\ell}\big) \qquad (\forall\,\ell\geqslant 1),$$

after division by ρ^ℓ, it would suffice to have, with certain new minorinzing functions:

$$g_{\ell,\rho} \ \leqslant \ \tfrac{1}{\rho^\ell}\, f_{\ell,\rho},$$

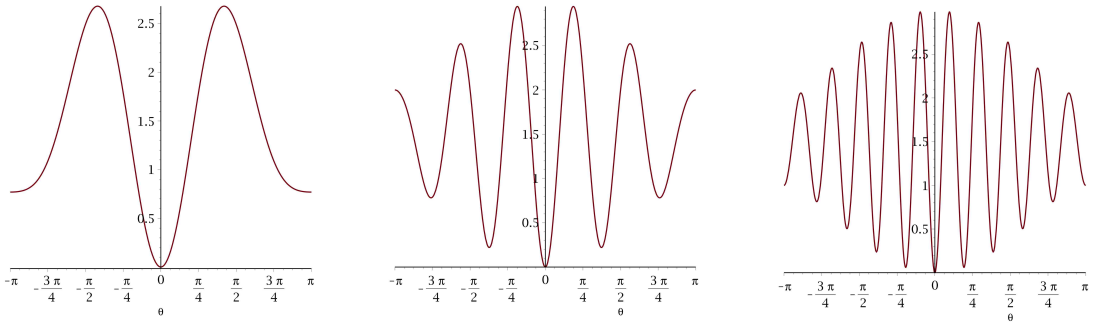
the nonnegativity:

$$\begin{aligned}
0 \ &\overset{?}{\leqslant} \ g_{\ell,\rho}(\theta) \\
&:= \ 1 - \cos\ell\theta + 2\,\rho\,\big(1 - \cos\,(\ell+1)\theta\big) \\
&\quad + \rho^{\ell+1}\,\Big[4\cos\theta + 4\cos\,(\ell+1)\theta - 4\cos\ell\theta - 4\Big] \\
&\quad + \rho^{2\ell+1}\,\Big[\,-\,8\cos\theta - 2\cos\,(\ell+1)\theta + 8\cos\ell\theta + 2\Big] \\
&\quad + \rho^{2\ell+2}\,\Big[2\cos\ell\theta - 2\Big] + \rho^{3\ell+1}\,\Big[4\cos\theta - 4\Big].
\end{aligned}$$

For instance, again with the choice $\rho := 0.25$, the graphs on the unit circle of the functions

$$\theta \ \longmapsto \ g_{\ell,\rho}(\theta) \qquad (-\pi\leqslant\theta\leqslant\pi)$$

show up, respectively, for the three choices $\ell = 2, 5, 10$, as:



Since these functions $g_{\ell,\rho}$ are even, it suffices to establish their nonnegativity on $[0,\pi]$. Let us begin with examining their behavior in a right half-neighborhood of 0. Starting from:

$$0 \ = \ g_{\ell,\rho}(0),$$

a positivity of the first derivatives of the $g_{\ell,\rho}$ would be welcome, at least on a small interval like $\big]0,\ \frac{\pi}{4\ell}\big]$.

Lemma 11.2. *For all real $0 < \rho \leqslant 0.25$ and for every integer $\ell \geqslant 1$, one has:*

$$g'_{\ell,\rho}(\theta) \ > \ 0 \qquad (\forall\ 0<\theta\leqslant\tfrac{\pi}{4\ell}).$$

Proof. Observe that this is true even when $\rho = 0$, since the function $g_{\ell,0}(\theta) = 1 - \cos\ell\theta$ has derivarive $\ell\sin\ell\theta > 0$ on $\big]0,\ \frac{\pi}{4\ell}\big]$. Anyway, we assume $0 < \rho \leqslant 0.25$.

Our aim is to minorize this derivative:

$$\begin{aligned} g'_{\ell,\rho}(\theta) &= \ell \sin \ell\theta + 2\,\rho\,(\ell+1)\sin(\ell+1)\theta \\ &\quad + \rho^{\ell+1}\Big[-4\sin\theta - 4(\ell+1)\sin(\ell+1)\theta + 4\ell\sin\ell\theta\Big] \\ &\quad + \rho^{2\ell+1}\Big[8\sin\theta + 2(\ell+1)\sin(\ell+1)\theta - 8\ell\sin\ell\theta\Big] \\ &\quad + \rho^{2\ell+2}\Big[-2\ell\sin\ell\theta\Big] + \rho^{3\ell+1}\Big[-4\sin\theta\Big], \end{aligned}$$

by a quantity which can be seen to be positive. However, we have to treat the special case $\ell = 1$ separately, namely for all $0 < \rho \leqslant \frac{1}{4}$ and for all $0 < \theta \leqslant \frac{\pi}{4}$, we first check that:

$$\begin{aligned} g'_{1,\rho}(\theta) &= \sin\theta + 4\rho\sin 2\theta + \rho^2\big[-8\sin 2\theta\big] + \rho^3\big[4\sin 2\theta\big] + \rho^4\big[-6\sin\theta\big] \\ &= \sin\theta\Big\{1 + 8\rho\cos\theta - \rho^2\,16\cos\theta + \rho^3\,8\cos\theta - \rho^4\,6\Big\} \\ &\geqslant \sin\theta\Big\{\underline{1}_{\circ} + 8\,\rho\,\tfrac{1}{\sqrt{2}} - \underline{\tfrac{1}{4^2}\,16}_{\circ} + \rho^3\,\tfrac{8}{\sqrt{2}} - \rho^4\,6\Big\} \\ &\geqslant \rho\sin\theta\cdot\Big\{\tfrac{8}{\sqrt{2}} + 0^2\,\tfrac{8}{\sqrt{2}} - \tfrac{1}{4^3}\,6\Big\} \\ &= \rho\sin\theta\cdot 5,563\cdots \\ &> 0. \end{aligned}$$

So we may assume $\ell \geqslant 2$. If we use the classical inequalities valid for $\varphi \in [0,\pi]$:

$$\sin\varphi \;\geqslant\; \varphi - \tfrac{1}{6}\,\varphi^3 \qquad\qquad \text{and} \qquad\qquad -\sin\varphi \;\geqslant\; -\varphi,$$

we are conducted to ask ourselves whether:

$$\begin{aligned} g'_{\ell,\rho}(\theta) &\geqslant \ell\,\big(\ell\theta - \tfrac{1}{6}(\ell\theta)^3\big) + 2\,\rho\,(\ell+1)\,\big((\ell+1)\theta - \tfrac{1}{6}((\ell+1)\theta)^3\big) \\ &\quad + \rho^{\ell+1}\Big[-4\theta - 4(\ell+1)\,(\ell+1)\theta + 4\ell\,\big(\ell\theta - \tfrac{1}{6}(\ell\theta)^3\big)\Big] \\ &\quad + \rho^{2\ell+1}\Big[8\,\big(\theta - \tfrac{1}{6}\theta^3\big) + 2\,(\ell+1)\,\big((\ell+1)\theta - \tfrac{1}{6}((\ell+1)\theta)^3\big) - 8\ell\,\ell\theta\Big] \\ &\quad + \rho^{2\ell+2}\big[-2\ell\,\ell\theta\big] + \rho^{3\ell+1}\big[-4\,\theta\big] \\ &\overset{?}{>} 0 \qquad\qquad (\forall\ 0<\theta\leqslant\tfrac{\pi}{4\ell}). \end{aligned}$$

To have a better view, let us set:

$$t := \ell\,\theta, \qquad\qquad \text{whence} \qquad\qquad 0 < t \leqslant \tfrac{\pi}{4} < 1,$$

and let us simplify this minorant by writing $(\ell+1)\theta = \frac{\ell+1}{\ell}\,\ell\theta = \frac{\ell+1}{\ell}\,t$:

$$\begin{aligned} g'_{\ell,\rho}(\theta) &\geqslant \ell\,t\,\big(1 - \tfrac{1}{6}\,t^2\big) + 2\,\rho\,(\ell+1)\,\tfrac{\ell+1}{\ell}\,t\big(1 - \tfrac{1}{6}\big(\tfrac{\ell+1}{\ell}\big)^2\,t^2\big) \\ &\quad + \rho^{\ell+1}\Big[-\tfrac{4}{\ell}\,t - 4(\ell+1)\,\tfrac{\ell+1}{\ell}\,t + 4\ell\,t\,\big(1 - \tfrac{1}{6}\,t^2\big)\Big] \\ &\quad + \rho^{2\ell+1}\Big[8\,\tfrac{t}{\ell}\,\big(1 - \tfrac{1}{6}\,\big(\tfrac{t}{\ell}\big)^2\big) + 2\,(\ell+1)\,\tfrac{\ell+1}{\ell}\,t\,\big(1 - \tfrac{1}{6}\big(\tfrac{\ell+1}{\ell}\big)^2\,t^2\big) - 8\ell\,t\Big] \\ &\quad + \rho^{2\ell+2}\big[-2\ell\,t\big] + \rho^{3\ell+1}\big[-\tfrac{4}{\ell}\,t\big] \\ &\overset{?}{>} 0 \qquad\qquad (\forall\ 0<\theta\leqslant\tfrac{\pi}{4\ell}). \end{aligned}$$

In order to minorize this by an even simpler quantity, we can use, since $\ell \geqslant 2$:

$$-\left(\tfrac{\ell+1}{\ell}\right)^2 \geqslant -\left(\tfrac{3}{2}\right)^2 \qquad \text{and also} \qquad -t^2 \geqslant -1,$$

so that:

$$\begin{aligned}\text{preceding minorant} \;\geqslant\; & \ell\, t\, \tfrac{5}{6} + \underline{2\,\rho\,(\ell+1)\,\tfrac{\ell+1}{\ell}\, t\, \tfrac{5}{8}} \\ & + \rho^{\ell+1}\Big[-2\,t - 4\,(\ell+1)\,\tfrac{3}{2}\,t + 4\,\ell\, t\,\tfrac{5}{6}\Big] \\ & + \rho^{2\ell+1}\Big[8\cdot 0 + 2\,(\ell+1)\,1\,t\,\tfrac{5}{8} - 8\,\ell\, t\Big] \\ & + \rho^{2\ell+2}\big[-2\,\ell\, t\big] + \rho^{3\ell+1}\big[-2\,t\big] \qquad \overset{?}{>}\; 0,\end{aligned}$$

and we even once more minorize this intermediate minorant by neglecting the term underlined and summing the expressions in brackets:

$$\begin{aligned} g'_{\ell,\rho}(\theta) \;\geqslant\; & \ell\, t\, \tfrac{5}{6} + 0 \\ & + \rho^{\ell+1}\Big[-8\,t + \tfrac{16}{6}\,\ell\, t\Big] \\ & + \rho^{2\ell+1}\Big[\tfrac{5}{4}\,t - \tfrac{27}{4}\,\ell\, t\Big] \\ & + \rho^{2\ell+2}\big[-2\,\ell\, t\big] + \rho^{3\ell+1}\big[-2\,t\big] \qquad \overset{?}{>}\; 0. \end{aligned}$$

We conclude by a factorization and by a final computer check, still for all $\ell \geqslant 2$:

$$\begin{aligned} g'_{\ell,\rho}(\theta) \;&\geqslant\; t\left\{\tfrac{5}{6}\,\ell - \rho^{\ell+1}\big[8 + \tfrac{16}{6}\,\ell\big] - \rho^{2\ell+1}\big[\tfrac{5}{4} + \tfrac{27}{4}\,\ell\big] - \rho^{2\ell+2}\big[2\,\ell\big] - \rho^{3\ell+1}\big[2\big]\right\} \\ &\geqslant\; t\left\{\tfrac{5}{6}\,\ell - 0.25^{\ell+1}\big[8 + \tfrac{16}{6}\,\ell\big] - 0.25^{2\ell+1}\big[\tfrac{5}{4} + \tfrac{27}{4}\,\ell\big] - 0.25^{2\ell+2}\big[2\,\ell\big] - 0.25^{3\ell+1}\big[2\big]\right\} \\ &\geqslant\; t\cdot 1,442\cdots. \end{aligned}$$

□

In summary, we have established for all $\ell \geqslant 1$ the positivity on a starting interval:

$$0 \;<\; g_{\ell,\rho}(\theta) \qquad (\forall\, \theta \in]0, \tfrac{\pi}{4\ell}]),$$

and our next goal is to establish the positivity of this minoring function $g_{\ell,\rho}$ on the remaining (large) subinterval of $[0, \pi]$:

$$0 \;\overset{?}{<}\; g_{\ell,\rho}(\theta) \qquad (\forall\, \theta \in [\tfrac{\pi}{4\ell}, \pi]).$$

We first finish the case $\ell = 1$.

Lemma 11.3. *For all $0 < \rho \leqslant 0.25$, the function $g_{1,\rho}(\theta)$ is positive on $\big[\frac{\pi}{4\cdot 1}, \pi\big]$.*

Proof. Indeed:

$$\begin{aligned} g_{1,\rho}(\theta) &= 1 - \cos\theta + 2\rho\,\big(1-\cos 2\theta\big) \\ &\quad + \rho^2\,\big[4\cos 2\theta - 4\big] \\ &\quad + \rho^3\,\big[-2\cos 2\theta + 2\big] \\ &\quad + \rho^4\,\big[6\cos\theta - 6\big] \\ &= \big(1-\cos\theta\big)\,\big[1-6\,\rho^4\big] + \big(1-\cos 2\theta\big)\,\big[2\rho\,\big(1-2\,\rho+\rho^2\big)\big] \\ &\geqslant \big(1-\tfrac{1}{\sqrt{2}}\big)\,\big[1-6\cdot 0.25^4\big] + \text{nonnegative} \\ &= 0.286\cdots \\ &> 0. \end{aligned}$$

□

From now one, when we work on $\big[\frac{\pi}{4\ell},\pi\big]$, we can therefore assume that:

$$\ell \geqslant 2.$$

Lemma 11.4. *For all $0<\rho<0.25$ and every integer $\ell\geqslant 2$, there is on $\big[\frac{\pi}{4\ell},\pi\big]$ a minoration:*

$$0 \overset{?}{<} h_{\ell,\rho}(\theta) \leqslant g_{\ell,\rho}(\theta) \leqslant \tfrac{1}{\rho^\ell}\,f_{\ell,\rho}(\theta),$$

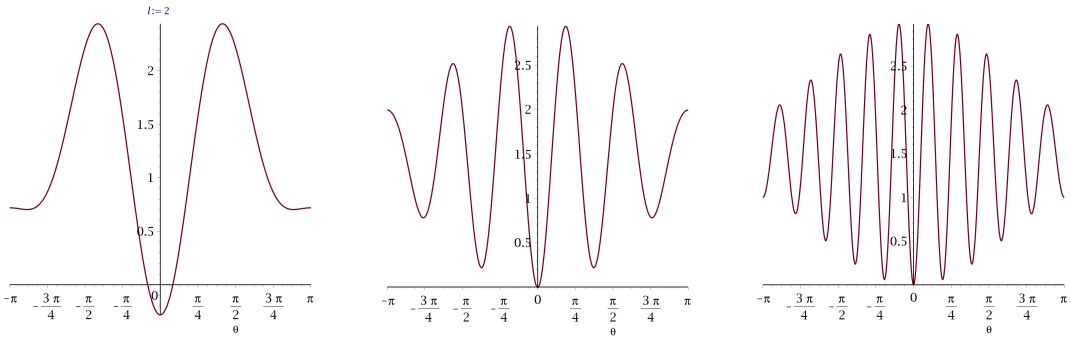
with the new:

$$h_{\ell,\rho}(\theta) := 1 - \cos\ell\theta + 2\rho\,\big(1-\cos(\ell+1)\theta\big) - 18\,\rho^{\ell+1}.$$

Again with the choice $\rho := 0.25$, the graphs on the unit circle of the functions

$$\theta \longmapsto h_{\ell,\rho}(\theta) \qquad (-\pi\leqslant\theta\leqslant\pi)$$

show up, respectively, for the three choices $\ell=2,5,10$, as:



Proof. Indeed, we minorize simply the reminders:

$$\begin{aligned} g_{\ell,\rho}(\theta) &\geqslant 1 - \cos\ell\theta + 2\rho\,\big(1-\cos(\ell+1)\theta\big) \\ &\quad - \rho^{\ell+1}\,\big[4+4+4+4\big] - \rho^{2\ell+1}\,\big[8+2+8+2\big] - \rho^{2\ell+2}\,\big[2+2\big] - \rho^{3\ell+1}\,\big[4+4\big], \end{aligned}$$

and to even simplify the second line by replacing it by $-18\,\rho^{\ell+1}$ as announced, we assert that:

$$-16\,\rho^{\ell+1} - 20\,\rho^{2\ell+1} - 4\,\rho^{2\ell+2} - 8\,\rho^{3\ell+1} \;\geqslant\; -18\,\rho^{\ell+1},$$

simply since:

$$\begin{aligned}\rho^{\ell+1}\left(1 - 10\,\rho^{\ell} - 2\,\rho^{\ell+1} - 4\,\rho^{2\ell}\right) &\geqslant \rho^{\ell+1}\left(1 - 10\cdot 0.25^2 - 2\cdot 0.25^3 - 4\cdot 0.25^4\right)\\ &> \rho^{\ell+1}\left(0.328\cdots\right)\\ &> 0.\end{aligned}$$

□

It therefore remains to treat all the cases $\ell \geqslant 2$. We start by looking at the subintervals $\left[\frac{\pi}{4\ell},\, \frac{7\pi}{4\ell}\right] \subset \left[\frac{\pi}{4\ell}, \pi\right]$.

Lemma 11.5. *For every real* $0 < \rho \leqslant 0.25$ *and every integer* $\ell \geqslant 2$*:*

$$0 \,<\, h_{\ell,\rho}(\theta) \qquad (\forall\ \theta \in [\tfrac{\pi}{4\ell}, \tfrac{7\pi}{4\ell}]).$$

Proof. Since:

$$\tfrac{\pi}{4} \,\leqslant\, \ell\,\theta \,\leqslant\, \tfrac{7\pi}{4}$$

it comes:

$$1 - \cos \ell\theta \,\geqslant\, 1 - \frac{1}{\sqrt{2}},$$

and since $1 - \cos(\ell+1)\theta \geqslant 0$ anyway, we can minorize:

$$\begin{aligned}h_{\ell,\rho}(\theta) &\geqslant 1 - \cos \ell\theta + 0 - 18\,\rho^{\ell+1}\\ &\geqslant 1 - \tfrac{1}{\sqrt{2}} - 18\cdot 0.25^{2+1}\\ &= 0.01164\cdots.\end{aligned}$$

□

We can now finish the case $\ell = 2$. It remains to show positivity of $h_{2,\rho}(\theta)$ on $\left[\frac{7\pi}{8}, \pi\right]$. Since $\frac{21\pi}{8} \leqslant 3\,\theta \leqslant 3\,\pi$, or equivalently $\frac{5\pi}{8} \leqslant 3\,\theta - \pi \leqslant \pi$, we can minorize:

$$\begin{aligned}h_{2,\rho}(\theta) &= 1 - \cos 2\theta + 2\rho\left(1 - \cos 3\theta\right) - 18\,\rho^3\\ &\geqslant 0 + 2\rho\left[\left(1 - \cos \tfrac{5\pi}{8}\right) - 9\cdot 0.25^2\right]\\ &= 2\rho\cdot 0.820\cdots\\ &> 0.\end{aligned}$$

It still remains to treat all the cases $\ell \geqslant 3$.

Lemma 11.6. *For every real* $0 \leqslant \rho \leqslant 0.25$ *and every integer* $\ell \geqslant 3$*, the function:*

$$h_{\ell,\rho}(\theta) \,:=\, 1 - \cos \ell\theta + 2\,\rho\left(1 - \cos(\ell+1)\theta\right) - 18\,\rho^{\ell+1}$$

takes only positive values in the interval $\left[\frac{7\pi}{4\ell}, \pi\right]$*:*

$$h_{\ell,\rho}(\theta) \,>\, 0 \qquad (\forall\ \tfrac{7\pi}{4\ell} \leqslant \theta \leqslant \pi).$$

Proof. Using $1 - \cos\varphi = 2\sin^2\frac{\varphi}{2}$, let us rewrite:

$$h_{\ell,\rho}(\theta) \,=\, 2\sin^2\tfrac{\ell\theta}{2} + 4\,\rho\sin^2\tfrac{(\ell+1)\,\theta}{2} - 18\,\rho^{\ell+1}.$$

At a point $\theta \in \left[\frac{7\pi}{4\ell}, \pi\right]$, if we have either:

$$2\sin^2\tfrac{\ell\theta}{2} - 18\,\rho^{\ell+1} \,>\, 0 \qquad \text{or} \qquad 4\,\rho\sin^2\tfrac{(\ell+1)\,\theta}{2} - 18\,\rho^{\ell+1} \,>\, 0,$$

then there is nothing to prove. We claim that the opposite inequalities cannot hold.

Assertion 11.7. *For every* $\ell \geqslant 3$*, there is no* $\theta \in \left[\frac{7\pi}{4\ell}, \pi\right]$ *at which:*

$$\sin^2\tfrac{\ell\theta}{2} \,\leqslant\, 9\,\rho^{\ell+1} \qquad \textit{and} \qquad \sin^2\tfrac{(\ell+1)\,\theta}{2} \,\leqslant\, \tfrac{9}{2}\,\rho^{\ell}.$$

Proof. Suppose nevertheless that such a $\theta \in \left[\frac{7\pi}{4\ell}, \pi\right]$ exists. Modulo π, there exist two unique representatives $-\frac{\pi}{2} < \alpha, \beta \leqslant \frac{\pi}{2}$ of:

$$\frac{\ell\,\theta}{2} - p\,\pi \,=\, \alpha \qquad \text{and} \qquad \frac{(\ell+1)\,\theta}{2} - q\,\pi \,=\, \beta,$$

with certain unique integers $p, q \in \mathbb{Z}$, whence:

$$\sin^2\alpha \,=\, \sin^2\tfrac{\ell\,\theta}{2} \,\leqslant\, 9\,\rho^{\ell+1} \qquad \text{and} \qquad \sin^2\beta \,=\, \sin^2\tfrac{(\ell+1)\,\theta}{2} \,\leqslant\, \tfrac{9}{2}\,\rho^{\ell}.$$

Lemma 11.8. *For all $0 \leqslant |\gamma| \leqslant \frac{\pi}{2}$, one has the classical inequality $\frac{|\gamma|}{2} \leqslant |\sin\gamma| \leqslant |\gamma|$.* □

Consequently, using $\rho^{1/2} \leqslant 0.25^{1/2} = \frac{1}{2}$, it comes:

$$\tfrac{|\alpha|}{2} \,\leqslant\, |\sin\alpha| \,\leqslant\, 3\,\rho^{\frac{\ell+1}{2}} \,\leqslant\, 3\,\tfrac{1}{2^{\ell+1}} \qquad \text{and} \qquad \tfrac{|\beta|}{2} \,\leqslant\, |\sin\beta| \,\leqslant\, \tfrac{3}{\sqrt{2}}\,\rho^{\frac{\ell}{2}} \,\leqslant\, \tfrac{3}{\sqrt{2}}\,\tfrac{1}{2^{\ell}},$$

that is to say:

$$|\alpha| \,\leqslant\, 3\,\tfrac{1}{2^{\ell}} \qquad \text{and} \qquad |\beta| \,\leqslant\, 3\sqrt{2}\,\tfrac{1}{2^{\ell}}.$$

From a chain of estimations:

$$\begin{aligned} 3\,\tfrac{1+\sqrt{2}}{2^{\ell}} \,\geqslant\, |\beta-\alpha| \,&=\, \left|\tfrac{\theta}{2} + (q-p)\,\pi\right| \\ &\geqslant\, \left|\tfrac{1}{2}\,\tfrac{7\pi}{4\ell} + (q-p)\,\pi\right| \\ &\geqslant\, \min_{r\in\mathbb{Z}} \left|\tfrac{7\pi}{8\ell} - r\,\pi\right| \\ [\tfrac{7\pi}{8\ell} \leqslant \tfrac{7\pi}{64}] \qquad &=\, \tfrac{7\pi}{8\ell} \end{aligned}$$

we obtain the inequality:

$$3\,\tfrac{1+\sqrt{2}}{2^{\ell}} \,\geqslant\, \tfrac{7\pi}{8\ell},$$

which is visibly false for large ℓ, and which begins to be false when $\ell \geqslant 3$:

$$3\,\tfrac{1+\sqrt{2}}{2^{3}} \,=\, 0.905\cdots \overset{\textbf{no!}}{\geqslant} 0.916\cdots \,=\, \tfrac{7\pi}{8\cdot 3},$$

This contradiction proves the assertion. □

The proof of Lemma 11.6 is complete. □

The proof of Proposition 11.1 is complete. □

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