THE DIMENSION OF THE MODULI SPACE OF SOLUTIONS OF THE GRAVITATIONAL MONOPOLE EQUATIONS VIA THE ATIYAH-SINGER INDEX THEOREMS

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Abstract. Our main Theorem is we determined via the Atiyah-Singer index theorem that the dimensions of the moduli space of solutions of the Gravitational monopole equations is

$$c_1(\sqrt{L})^2 + \frac{57\sigma(X) - 30\chi(X)}{4}. \tag{0.1}$$

1. Introduction

It had been observed in [cf.2] and [cf.3] and other references that in four-dimensions, Einstein's equations with a nonzero cosmological constant ($R_{ij} = \lambda g_{ij}$, λ is the cosmological constant) on a Euclidean self-dual spin manifold M can be replaced by five quadratic conditions on the curvature of an $SU(2)$-spin connection, namely

$$\frac{1}{4}F_{[ab}{}^{(AB}F_{cd]}^{CD)} = 0. \tag{1.1}$$

The F_{ab} are the curvature components of the self-dual spin connection ω_a^+ on the principal $SU(2)$-bundle $SU(2) \to P \xrightarrow{M}$.

According to [cf.3], equations (1.1) is equivalent to the following set of equations,

$$F_{ab}{}^{AB} = -\frac{1}{6}\lambda\Sigma_{ab}{}^{AB} \tag{1.2}$$

where

$$\Sigma_{ab}{}^{AB} := 2\gamma_{[a}{}^{AA'}\gamma_{b]}{}_{A'}{}^{B}, \tag{1.3}$$

where γ_a is an $SU(2) \times SU(2)$ soldering form which defines a metric in the following form

$$g_{ab} = \gamma_a{}^{AA'}\gamma_{bAA'}; \tag{1.4}$$

with respect to this metric the Hodge $*$-operator in 4-dimensions helps to conclude Σ_{ab} is self-dual, using the representation (1.2), one also concludes F_{ab} is self-dual. Fixing the orientation of the manifold, one can conclude, the anti-self-dual part of the Weyl tensor vanishes, namely

$$W_- = 0, \tag{1.5}$$

therefore the 4-manifold is self-dual. Conversely, any conformally self-dual Einstein manifold with a cosmological constant (the constant can be zero, then it will only be a solution of vanishing Ricci tensor, namely $R_{ij} = 0$) arises as a solution the equation (1.1).

The author in a previous preprint [cf.4] had shown if the 4-manifold allows a Gravitational monopole equation [cf.5], then the strict inequality $\lambda < 0$ occurs.

It is shown in [cf.2] that the linearized version of (1.1) has the following form with respect to a perturbation $A \mapsto A + C$

$$\mathbb{D}_1 C := F_{[ab}{}^{(AB}D_c C_{d]}{}^{CD)} = 0. \tag{1.6}$$

Here D_a is a covariant derivative with respect to A with curvature F. Infinitesimal gauge transforms correspond to perturbations of the form [cf.2]

$$C_a := \mathbb{D}_0(f, M, N) := (\nabla^b f)F_{ba} + [D^b M, F_{ba}] + D_a N, \tag{1.7}$$

where $N, M \in \mathfrak{su}(2)$ and f is a positive real-valued smooth function. One can show

$$C \in \ker \mathbb{D}_1 / \operatorname{Im} \mathbb{D}_0 \tag{1.8}$$

One notes, neither $\mathbb{D}_0$ nor $\mathbb{D}_1$ are elliptic, therefore one needs to "enlarge" the domain of the operator, and one defines

$$\begin{aligned} \mathbb{D} &= (\mathbb{D}, \mathbb{D}_0^\dagger) \\ \mathbb{D}_0^\dagger &= (\operatorname{Tr} F^{ab} D_a C_b; [D_a C_b, F^{ab}]; -D^a C_a) \end{aligned} \tag{1.9}$$

where the adjoint $\mathbb{D}_0^\dagger$ of $\mathbb{D}_0$ is defined with respect to the inner-product

$$\langle C, \overline{C} \rangle = \int_M \sqrt{g} g^{ab} C_a{}^{AB} \overline{C}_b{}_{AB}. \tag{1.10}$$

One can rephrase the above computations in terms of the following chain complex

$$0 \to \wedge^0 \otimes \wedge^0 \otimes \wedge^0 \xrightarrow{\mathbb{D}_0} \wedge^1 \xrightarrow{\mathbb{D}_1} \wedge^4 \tag{1.11}$$

such that

$$\mathbb{D}_1 \mathbb{D}_0 = 0. \tag{1.12}$$

More explicitly [cf. 2]

$$\begin{aligned} \mathbb{D}_0^* &: \wedge^1 \to \wedge^0 \otimes \wedge^0 \otimes \wedge^0, \\ \mathbb{D}_0^* C &= \big(\operatorname{Tr} F^{ab} D_a C_b; [D_a C_b, F^{ab}]; -D^a C_a\big), \\ \mathbb{D}_1^* &: \wedge^4 \to \wedge^1, \mathbb{D}_1^* \omega = F_{CD}^{cd} D^b \omega_{abcd}^{ABCD} \end{aligned} \tag{1.13}$$

The covariant derivative also acts on tensors and can be assumed to be torsion free and is compatible with respect to the metric obtained from equation (1.4). One therefore can construct corresponding "Laplacians" that act on the sections of the bundles in question [cf.2]

$$\begin{aligned} \Delta_0 : \wedge^0 \otimes \wedge^0 \otimes \wedge^0 \to \wedge^0 \otimes \wedge^0 \otimes \wedge^0, &\quad \Delta_0 := \mathbb{D}_0^* \mathbb{D}_0; \\ \Delta_1 : \wedge^1 \to \wedge^1, &\quad \Delta_1 := \mathbb{D}_1^* \mathbb{D}_1 + \mathbb{D}_0 \mathbb{D}_0^*; \\ \Delta_2 : \wedge^4 \to \wedge^4, &\quad \Delta_2 := \mathbb{D}_1 \mathbb{D}_1^*. \end{aligned} \tag{1.14}$$

According to the Fredholm alternative, these Laplacians have an orthogonal decomposition of the space of sections of each vector bundle, therefore [cf.2]

$$\begin{aligned} \wedge^0 \otimes \wedge^0 \otimes \wedge^0 &\cong \operatorname{range}(\mathbb{D}_0^*) \oplus \ker \Delta_0, \\ \wedge^1 &\cong \operatorname{range}(\mathbb{D}_0) \oplus \operatorname{range}(\mathbb{D}_1^*) \oplus \ker \Delta_1, \\ \wedge^4 &\cong \operatorname{range}(\mathbb{D}_1) \oplus \ker \Delta_2 \end{aligned} \tag{1.15}$$

where the orthogonality of the decomposition is with respect to the inner products described above. One can show, as in the case of de Rham cohomology, that the equivalence class $[C]$, defined in (1.8), can be identified with the kernel of the Laplacian (1.14) on Lie-algebra valued one-forms [cf. 2]:

$$\ker \mathbb{D}_1 / \operatorname{Im} \mathbb{D}_0 \cong \ker \Delta_1. \tag{1.16}$$

One uses the definition of the Laplacians as in (1.14), and verifies explicitly that the perturbations are elements of kernel of Δ_1 if and only if

$$\mathbb{D}_1 C = 0 = \mathbb{D}_0^* C. \tag{1.17}$$

these equations can be viewed as a combination of the linearized instanton equation and "gauge fixing" conditions [cf. 2].

Now the linearized instanton equation (1.6) is equivalent to the following equation [cf. 2]

$$\mathbb{D}_1^* \mathbb{D}_1 C = 0, \tag{1.18}$$

this implies,

$$D^b (F_{abAB} F^{cd(AB} D_c C_d{}^{CD)}) = 0. \tag{1.19}$$

From (1.17) we also get,

$$\begin{aligned} \operatorname{Tr} F^{ab} D_a C_b &= 0, \\ [F^{ab}, D_a C_b] &= 0, \\ D^a C_a &= 0. \end{aligned} \tag{1.20}$$

First two equations of (1.20) implies [cf. 2]

$$F^{abAB} D_a C_b{}^{CD} = F^{ab(AB} D_a C_b^{CD)}, \tag{1.21}$$

So, we remove the symmetrization in (1.19).

Since [cf. 2]

$$\Sigma_{ab}{}^{AB} \Sigma_{AB}^{cd} = 4(\delta_a^{[c} \delta_b{}^{d]} + \frac{1}{2} \epsilon_{ab}^{cd}). \tag{1.22}$$

we can replace (1.19) by

$$D^b [(\delta_a{}^{[c} \delta_b{}^{d]} + \frac{1}{2} \epsilon_{ab}^{cd}) D_c C_d] = 0. \tag{1.23}$$

One can expand the equation (1.23) and finds terms involving a gauge-co variant Laplacian, the Ricci tensor, the Riemann tensor, the $SU(2)$ curvature, and the gradient of a divergence; the latter three of these can be made to vanish by using the cyclic identity for the Riemann tensor, the self-duality of the $SU(2)$ curvature, and the last equation of (1.20), respectively [cf. 2]. One therefore can deduce that the equation (1.17) implies

$$-D^a D_a C_b + R_b{}^a C_a = 0, \tag{1.24}$$

where $R_b{}^a$ is the Ricci tensor. We know, if M is Einstein, then

$$R_b{}^a = \lambda \delta_a{}^b. \tag{1.25}$$

One uses (1.24) and (1.25) to get

$$(-D^a D_a + \lambda) C_b = 0. \tag{1.26}$$

Equation (1.26) is an elliptic second-order differential equation [cf. 2] and, can be shown using the above arguments that the kernel of an e11iptic operator on a compact manifold, as a vector space over the reals, is finite-dimensional, the physical instanton perturbations form a finite-dimensional subspace of all possible gravitational perturbations [cf. 2].

Usually the linear equation (1.6) equation has infinitely many solution but modulo the gauge group, the solution space of the elliptic operator $\mathbb{D}$ is finite dimensional when M is compact and without boundary (closed), this however means that the gauge-inequivalent solutions form a finite-dimensional subspace of all possible perturbations.

Since, $\lambda < 0$ after introduction of the Gravitational monopole equation, one may determine the dimension of the moduli space of solutions using the Atiyah-Singer index theorem. It can be shown that the linearization is *stable* if the kernel of the adjoint operator of the elliptic operator $\mathbb{D}$ is trivial, more precisely,

$$\mathbb{D}^* = (\mathbb{D}_1^\dagger, \mathbb{D}_0) \tag{1.27}$$

has a trivial kernel if the perturbation is stable, here $\mathbb{D}_1^*$ is the L^2 adjoint with respect to the inner-product introduced above (2.3). Point to be noted is, $\mathbb{D}_1^*$ acts on totally symmetric, valence-four spinor-valued four-forms ω

$$\mathbb{D}_1^\dagger \omega = F^{cd}{}_{CD} D^b \omega_{abcd}{}^{ABCD} \tag{1.28}$$

One can show the linearization (1.6) is stable if

$$\ker \mathbb{D}^\dagger = \ker \mathbb{D}_0 \cup \ker \mathbb{D}_1^\dagger \tag{1.29}$$

is trivial. If $\ker \mathbb{D}_0 \neq \{0\}$ the corresponding connection A is said to be *reducible*, that is there is an infinitesimal automorphism of $SU(2) \to P \to M$ fixes A. Now if $\ker \mathbb{D}_1^\dagger \neq \{0\}$, then corresponding non-zero spinor $\omega_{abcd}{}^{ABCD}$ satisfying

$$\mathbb{D}_1^\dagger \omega_{abcd}{}^{ABCD} = 0 \tag{1.30}$$

will be called *harmonic Weyl spinor type*, as $\omega_{abcd}{}^{ABCD}$ has all the symmetries of a Weyl spinor.

To make our Gravitational monopole solutions stable, we will always assume $\omega_{abcd}{}^{ABCD} = 0$. But this give rise to a new problem, we are forced to study Einstein spaces with harmonic Weyl spinor that allows Gravitational monopole equations, we keep in mind that $W_\pm \neq 0$.

In the following sections, we shall linearize $W_\pm$ and compute the dimension of moduli space of solutions of the Gravitational monopole equations.

2. Mathematical Preliminaries

Let M be smooth Riemannian manifold with a Riemannian metric g. Two metrics g_1 and g_2 are conformally equivalent if there is a smooth nonzero function $f : M \to \mathbb{R}_+$ to the positive reals such that $g_1 = f g_2$. A conformal structure c on M is an equivalence class of metrics $[g]$, such that $c := [g]$. In particular when $\dim_{\mathbb{R}} M = 4$, the Hodge isomorphism

$$*_g : \Omega^p(M) \to \Omega^{4-p}(M), \quad *_g^2 = 1, \tag{2.1}$$

induces a splitting

$$\Omega^2(M) \cong \Omega^+ \oplus \Omega^- \tag{2.2}$$

into the $\pm$ eigenspaces of $*_g$.

One introduces a bilinear form on Ω^2 using cup product, namely

$$\langle \alpha, \beta \rangle = (\alpha \cup \beta)[M]. \tag{2.3}$$

With respect to the bilinear form (2.3) Ω^+ (resp. Ω^-) is positive (negative) definite. Actually, any smooth 3-dimensional subbundle of Ω^2 with the stated property defines a unique conformal structure for which it coincides with either Ω^+ or Ω^-. Infinitesimal deformations of a conformal structure can be described by sections of the bundle $\mathrm{Hom}(\Omega^+, \Omega^-)$.

Let us consider the splitting corresponding to (2.2)

$$\mathfrak{so}(4) \cong \mathfrak{so}(3)^+ \oplus \mathfrak{so}(3)^- \tag{2.4}$$

of Lie algebras. Consequently, we have a splitting of the bundle $\mathfrak{so}(4)[M]$ of infinitesimal isometries of TM into $\mathfrak{so}(3)^+(M) \oplus \mathfrak{so}(3)^-(M)$. The Riemannian curvature tensor is $\Omega^2 \otimes \mathfrak{so}(4)(M)$-valued. Therefore, we can consider its component in $\Omega^\pm \otimes \mathfrak{so}^\pm(3)(M)$. The metric provides an isomorphism between the bundles TM and T^*M. In a similar manner the metric g defines an isomorphism

$$\Omega^2(M) \cong \mathfrak{so}(4)(M). \tag{2.5}$$

The isomorphism (2.5) provides a linear mapping of $\Omega^\pm(M)$ into $\mathfrak{so}(3)^\pm$. The construction induces a splitting $\Omega^\pm \otimes \mathfrak{so}(3)^\pm$ into a trace part, an antisymmetric part, and a trace free symmetric part. The trace free symmetric part is denoted by $\Omega^\pm \otimes_{Sym} \mathfrak{so}(3)^\pm$. The corresponding component of the curvature tensor is the self-dual (anti-self-dual) Weyl tensor W^+ (W^-).

Let us denote the linearization of $W^\pm$ by

$$D : C^\infty\big(\mathrm{Hom}(\Omega^\pm, \Omega^\mp)\big) \to \Omega^\pm \otimes_{Sym} \mathfrak{so}(3)^\pm. \tag{2.6}$$

It is a second-order differential operator. We used above a special property of the Weyl tensor in dimension four, is that it can be decomposed in an invariant way into two equal parts $W^{\pm}$. So, require a conformal structure c to satisfy only "half" of the integrability condition. We say that c is self-dual (anti-self-dual) if

$$W_{-}(c) = 0 \quad (W_{+}(c) = 0). \tag{2.7}$$

We need a fact about the self-duality (anti-self-duality) equation, that its linearization is an elliptic equation modulo the action of the diffeomorphism group of M. Consequently, the set

$$\mathcal{C}_{\pm}(M) = \{c \mid W_{\pm}(c) = 0\}/Diff(M) \tag{2.8}$$

of equivalence classes of self-dual (anti-self-dual) conformal structures, we expect them to be a smooth manifold whose dimension can be calculated by the Atiyah-Singer Index Theorem as the index of the corresponding elliptic system. One can show that this is the case under two additional conditions. The first condition can be stated in the following way: it is the vanishing of the cokernel of the linearization of W at $c \in \mathcal{C}_{\pm}(M)$. The second condition is that the group of diffeomorphisms of M acts freely on the space of self-dual (anti-self-dual) connections. One can weaken this condition by considering the dimension of the group G_c of conformal diffeomorphisms of c is (locally) constant at c. Then the Atiyah-Singer Index Theorem says

$$\dim \mathcal{C}_{\pm}(M) - \dim G_c = \mathrm{Index}(M) = \frac{1}{2}(29|\sigma| - 15\chi), \tag{2.9}$$

where σ is the signature of M, and χ is its Euler characteristic.

The linearized operation of the diffeomorphism group on $\mathcal{C}$ is described by the operator

$$L_{\pm} : C_0^{\infty}(TM) \to C_0^{\infty}\big(\mathrm{Hom}(\Omega^{\pm}, \Omega^{\mp})\big); \quad (LX)(\lambda) = \pi_{\mp} L_X \lambda, \tag{2.10}$$

where $(L_X)_{\pm}$ is the Lie derivative on the set of 2-forms, and $\pi_{\mp}$ is the projection onto $\Omega^{\mp}$. One can show $L_{\pm}$ is a differential operator in the vector field alone. Let us denote by $L^{\dagger}$ the L^2-adjoint with respect to the Riemannian metric g. One can show that $(D, L^{\dagger})$ is an elliptic system of partial differential equations (of mixed order). For a compact manifold M,, a smooth principal bundle P over M, $(D, L^{\dagger})$ induces a Fredholm operator

$$(D, L^{\dagger}) : U \to V \oplus W, \tag{2.11}$$

where, $U = L_2^p(\mathrm{Hom}(\Omega^+, \Omega^-))$, $V = L_1^p(TP)$, $W = L^p(\Omega^+ \otimes_{Sym} \mathfrak{so}(3)^+)$ are Sobolev spaces with respect to some metric on M.

3. Linearization of the Weyl tensor

The Weyl tensor in terms of indices is written the following form

$$W_{uijk} = R_{uijk} + \frac{1}{n-2}(R_{uk}g_{ij} - R_{uj}g_{ik} + R_{ij}g_{uk} - R_{ik}g_{uj}) + \frac{1}{(n-1)(n-2)}R(g_{uj}g_{ik} - g_{uk}g_{ij}). \tag{3.1}$$

where,

$$R^t_{ijk} = -\frac{\partial}{\partial x^k}\Gamma^t_{ij} + \frac{\partial}{\partial x^j}\Gamma^t_{ik} - \Gamma^s_{ij}\Gamma^t_{sk} + \Gamma^s_{ik}\Gamma^t_{sj}, \tag{3.2}$$

keeping in mind that $R_{uijk} = g_{ut}R^t_{ijk}$, and $\Gamma^u_{ij} := g^{uk}\Gamma_{ij,k} := \frac{1}{2}\Big(\frac{\partial}{\partial x^j}g_{ik} + \frac{\partial}{\partial x^i}g_{jk} - \frac{\partial}{\partial x^k}g_{ij}\Big)$, the notation $\Gamma_{ij,k}$ is the Christoffel symbol of the first kind, and Γ^k_{ij} or $\left\{ {k \atop ij} \right\}$ is the Christoffel symbol of the second kind, hence,

$$R_{uijk} = \frac{1}{2}\Big(\frac{\partial^2}{\partial x^i \partial x^j}g_{uk} + \frac{\partial^2}{\partial x^u \partial x^k}g_{ij} - \frac{\partial^2}{\partial x^u \partial x^j}g_{ik} - \frac{\partial^2}{\partial x^i \partial x^k}g_{uj}\Big) + g^{ts}(\Gamma_{ku,t}\Gamma_{ij,s} - \Gamma_{uj,t}\Gamma_{ik,s}). \tag{3.3}$$

The Ricci tensor is defined and denoted by

$$R_{ij} := R^t_{ijt} = \frac{\partial}{\partial x^j}\Gamma^t_{it} - \frac{\partial}{\partial x^t}\Gamma^t_{ij} + \Gamma^s_{it}\Gamma^t_{sj} - \Gamma^s_{ij}\Gamma^t_{st}$$
$$= \frac{\partial^2}{\partial x^i \partial x^j}\log\sqrt{|g|} - \frac{\partial}{\partial x^t}\Gamma^t_{ij} + \Gamma^s_{it}\Gamma^t_{js} - \Gamma^s_{ij}\frac{\partial}{\partial x^s}\log\sqrt{|g|}. \tag{3.4}$$

One notes $R_{ij} = R_{ji}$, the *curvature invariant* $R = g^{ij}R_{ij}$. A space is called an *Einstein space* if $R_{ij} = \lambda g_{ij}$, where λ is an invariant. We therefore have

$$R = n\lambda, \quad \text{and hence} \quad R_{ij} = \frac{R}{n}g_{ij}. \tag{3.5}$$

Now we expand the Weyl tensor around the perturbed flat metric η with "small" perturbation h with a dimensionless constant α,

$$g_{\mu\nu} = \eta_{\mu\nu} + \alpha h_{\mu\nu}. \tag{3.6}$$

The result is,

$$R^{lin}_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_\alpha\partial_\mu h_{\beta\nu} + \partial_\beta\partial_\nu h_{\alpha\mu} - \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\beta\partial_\mu h_{\alpha\nu}). \tag{3.7}$$

$$R^{lin}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\lambda}(\partial_\mu\partial_\alpha h_{\lambda\nu} - \partial_\lambda\partial_\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h_{\lambda\alpha} + \partial_\lambda\partial_\nu h_{\mu\alpha}) \tag{3.8}$$

$$R^{lin} = \eta^{\alpha\lambda}\eta^{\mu\nu}(\partial_\lambda\partial_\nu h_{\mu\alpha} - \partial_\alpha\partial_\nu h_{\mu\nu}). \tag{3.9}$$

Putting everything together in the Weyl tensor formula (3.1) we get the required linearisation.

3.1. **Linearization of Self-Dual (Anti-Self-Dual) Weyl tensors.** We recall that the linearization of $W^\pm$ is given by

$$D : C^\infty\big(\mathrm{Hom}(\Omega^\pm, \Omega^\mp)\big) \to \Omega^\pm \otimes_{Sym} \mathfrak{so}(3)^\pm. \tag{3.10}$$

It is a second-order differential operator. We can rewrite it into the following form:

$$\begin{aligned}
&\text{Linearization of } W_+ : \quad D^2 : \Gamma(\Omega^2_+ \otimes \Omega^2_-) = \Gamma(S^2_- \otimes S^2_+) \xrightarrow{\pi_+} \Gamma(S^4_-)\\
&\qquad\qquad \varphi_{A'B'AB} \mapsto \nabla^A_{(C'}\nabla^B_{D'}\varphi_{A'B')AB}\\
&\text{Linearization of } W_- : \quad D^2 : \Gamma(\Omega^2_- \otimes \Omega^2_+) = \Gamma(S^2_+ \otimes S^2_-) \xrightarrow{\pi_-} \Gamma(S^4_+)\\
&\qquad\qquad \varphi_{ABA'B'} \mapsto \nabla^{A'}_{(C}\nabla^{B'}_{D}\varphi_{AB)A'B'}
\end{aligned} \tag{3.11}$$

Let X be an oriented Riemannian manifold of even dimension $2l$ and we also assume X is a spin manifold, that is the first and second Stiefel–Whitney classes vanish. We denote by $\wedge^p$ the bundle of exterior p-forms with $A^p = \Gamma(\wedge^p)$ its space of smooth sections. The Hodge star operator $*\wedge^p \to \wedge^{2l-p}$is defined by,

$$\alpha \wedge *\beta = (\alpha, \beta)\omega \in \wedge^{2l} \tag{3.12}$$

where $\alpha, \beta \in \wedge^p$, (α, β) is the induced inner product on p-forms and ω is the volume form.

From now everything will be 4-dimensional unless otherwise stated. We start with the symmetry of the equations, namely the Lie algebras. The Lie algebra $\mathfrak{so}(4)$ of the special orthogonal group $SO(4)$ is not simple. It can be decomposed into the direct sum of two copies of the Lie algebra $\mathfrak{so}(3)$ of the group $SO(3)$:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3). \tag{3.13}$$

In terms of the group theory, one understands the above decomposition corresponds to the fact that the universal covering group of $SO(4)$ is the product of the two copies of $SU(2)$. This fact in quantum mechanics corresponds to $\pm\frac{1}{2}$ spins of an electron for each factor $SU(2)$.

In terms of the geometry of the vector bundles, the decomposition $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ induces the following decomposition (for a choice of g on X^4) for the vector bundle $\bigwedge^2 T^*X \to X$,

$$\wedge^2 T^*X \cong \wedge^+ \oplus \wedge^-, \tag{3.14}$$

as a Whitney sum of two oriented 3-plane bundles. One can choose an oriented orthonormal frame for $T^*_U X$ for an open set $U \subset X$. One therefore has,

$$\wedge^{\pm} = \mathrm{Span}\big\{(e^1 \wedge e^2 \pm e^3 \wedge e^4), (e^2 \wedge e^3 \pm e^1 \wedge e^4), (e^3 \wedge e^1 \pm e^2 \wedge e^4)\big\}. \tag{3.15}$$

We now use the unique Levi-Civita connection ∇ on $\wedge^2 T^*X$ to find a suitable decomposition of the curvature tensor under the action of $O(4)$. The first step towards it is to note that $\nabla g = 0$, this however means that ∇ is covariantly constant, that is ∇ maps sections of $\wedge^{\pm}$ into $\wedge^{\pm} \otimes T^*X$; there is no mixed term mapping $\wedge^+$ into $\wedge^- \otimes T^*X$. The curvature of the Levi-Civita connection defines a section of $\wedge^2 T^*X \otimes \bigwedge^2 T^*X$, correspondingly a decomposition of $\wedge^2 T^*X \otimes \bigwedge^2 T^*X$ into four matrix-blocks of size 3×3, more precisely, the Riemann curvature tensor defines, in general, a self-adjoint linear transformation $\mathcal{R} : \wedge^2 \to \wedge^2$ such that,

$$\mathcal{R}(e^i \wedge e^j) = \frac{1}{2} \sum_{k,l} R_{ijkl} e^k \wedge e^l, \tag{3.16}$$

relative to the decomposition $\wedge^2 = \wedge^+ \otimes \wedge^-$, the operator $\mathcal{R}$ has the following form,

$$\mathcal{R} = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix} \tag{3.17}$$

where, $B \in Hom(\wedge^-, \wedge^+)$ (is the traceless Ricci curvature) $\overset{0}{Ric}$, and $A \in \mathrm{End}(\wedge^+)$, that is A is symmetric about its diagonal, that is $A^t = A$, similarly for $C \in \mathrm{End}(\wedge^-)$ we have $C^t = C$.

This representation of the curvature tensor $\mathcal{R}$ gives us a complete decomposition of it into irreducible components, namely

$$\mathcal{R} \to (\mathrm{Tr}\, A, B, \underbrace{A - \frac{1}{3}\mathrm{Tr}\, A}_{W^+}, \underbrace{C - \frac{1}{3}\mathrm{Tr}\, C}_{W^-}) \tag{3.18}$$

$TrA = \mathrm{Tr}\, C = \frac{1}{4}s$ where s is the scalar curvature.

More elaborately, if the basis of $\wedge^{\pm}$ in (3.15) is denoted by $\{x^i_{\pm}\}_{i=1}^3$, then the curvature tensor $\mathcal{R}$ has the following expansion

$$\mathcal{R} = W^+_{ij} x^i_+ \otimes x^j_+ + W^-_{ij} x^i_- \otimes x^j_- + B_{ij} x^i_+ \otimes x^j_- + B^t_{ij} x^i_- \otimes x^j_+ - \frac{s}{12}(x^i_+ \otimes x^j_- + x^i_- \otimes x^j_+). \tag{3.19}$$

If we denote the projection operator by,

$$P_{\pm} := \frac{1}{2}(1 \pm *) : \wedge \to \wedge^{\pm}, \tag{3.20}$$

then,

$$W^{\pm} = P_{\pm} \circ Rm \circ P_{\pm} - \frac{s}{12} Id_{\pm} \tag{3.21}$$

We now assume $X \cong \mathbb{R} \times M^3$ with M^3 is smooth and has has non-positive scalar curvature. In a local *geodesic coordinate* the curvature tensor (3.3) has the following form

$$\begin{aligned}
\overset{0}{R}_{lijk} &= -\frac{\partial}{\partial x^l} \overset{0}{\Gamma}_{jk,i} + \frac{\partial}{\partial x^k} \overset{0}{\Gamma}_{jl,i} \\
\overset{0}{R}_{0jkl} &= \frac{\partial}{\partial x^l} \overset{0}{\Gamma}_{jk,0} - \frac{\partial}{\partial x^k} \overset{0}{\Gamma}_{jl,0} = \frac{1}{2}\Big[\frac{d}{dt}\Big(\frac{\partial}{\partial x^k} \overset{0}{g}_{jl}\Big) - \frac{d}{dt}\Big(\frac{\partial}{\partial x^l} \overset{0}{g}_{jk}\Big)\Big] \\
\overset{0}{R}_{0j0l} &= \frac{\partial}{\partial x^l} \overset{0}{\Gamma}_{j0,0} - \frac{\partial}{\partial x^0} \overset{0}{\Gamma}_{jl,0} = \frac{1}{2}\frac{d^2}{dt^2} \overset{0}{g}_{jl}
\end{aligned} \tag{3.22}$$

We use the "electric" and "magnetic" field on $\mathbb{R} \times M^3$ to linearize the self-dual Weyl tensor, this is how it is done, one has the isomorphism

$$\Omega^1(M^3) \oplus \Omega^1(M^3) \to \Omega^2(\mathbb{R} \times M^3) \tag{3.23}$$

via

$$\omega = dt \wedge \pi^*(E) + \pi^*(*B),$$
$$E_k = \omega_{0k}, \quad B_i = \frac{1}{2}\varepsilon_{ijk}\omega_{jk} \tag{3.24}$$

We now exploit the bi-linear nature of the curvature form R with respect to electric and the magnetic fields

$$\langle (E,B), R(E',B')\rangle = \alpha_{ij}E_iE'_j + \beta_{ij}(E_iB'_j + E'_iB_j) + \gamma_{ij}B_iB_j \tag{3.25}$$

and get back the decomposition (3.17) of the curvature tensor R. Therefore the Weyl tensor decomposition (3.21) implies, the linearisation of $W^{\pm}$ (upto the nonlinear terms) are the following

$$\alpha_{ij} + \beta_{ij} + \gamma_{ij} \tag{3.26}$$

in (3.17). More explicitly

$$\begin{aligned}
\alpha_{jl} &= R_{0j0l} = \frac{1}{2}\frac{d^2}{dt^2}\overset{0}{g}_{jl},\\
\beta_{ij} &= \frac{1}{2}\varepsilon_{jkl}R_{0ikl} = \frac{1}{4}\varepsilon_{jkl}\Big[\frac{d}{dt}\Big(\frac{\partial}{\partial x^k}\overset{0}{g}_{jl}\Big) - \frac{d}{dt}\Big(\frac{\partial}{\partial x^l}\overset{0}{g}_{jk}\Big)\Big] = -\frac{1}{2}\varepsilon_{jkl}\frac{d}{dt}\Big(\frac{\partial}{\partial x^k}\overset{0}{g}_{jl}\Big)\\
\gamma_{ij} &= \frac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}R_{klmn} = -(Ric_{ij} - \frac{1}{2}g_{ij}\operatorname{Tr} Ric) = -Ein_{ij},
\end{aligned} \tag{3.27}$$

Thus, the linearization of W^+ is, (dots denote derivatives in the $\mathbb{R}$-direction)

$$(W^+)_{ij} = \frac{1}{2}\frac{d^2}{dt^2}\overset{0}{g}_{jl} - \frac{1}{2}\varepsilon_{jkl}\frac{d}{dt}\Big(\frac{\partial}{\partial x^k}\overset{0}{g}_{jl}\Big) - Ein_{ij} = (\frac{1}{2}\ddot{g} - \frac{1}{2}d\dot{g} - Ein)_{ij}. \tag{3.28}$$

Thus, we get the linear operator modulo trace,

$$D = \frac{1}{2}\ddot{h} - \frac{1}{2}\not{d}h - \not{E}h. \tag{3.29}$$

We use this equation to compute the dimensions of the moduli space of solutions on the finite and infinite cylinders $[a,b]\times M^3$ and $\mathbb{R}\times M^3$. This will appear in a future work of the author [cf.1].

4. The Gravitational Monopole Equations

In [cf. 5] the Gravitational Monopole equations were introduced in the following sense. Let (X^4, g) be a Riemannian spin 4-manifold. Then the Clifford algebra bundle $\mathcal{C}l(X^4)$ is a vector bundle over X^4 with fibre at $x \in X^4$ is the Clifford algebra $\mathcal{C}l(T_xX)$. With respect to the metric g, one identifies (isomprphism) $\mathcal{C}l(T_xX)$ with $\mathcal{C}l(T^*_xX)$. Therefore, as a vector space, this is isomorphic to $\wedge T^*_xX$. Let us also assume $E \to X$ is a Clifford module bundle with a covariant derivative ∇^E. Then for each $x \in X$ there is a Clifford action $c : T^*_xX \otimes E_x \to E_x$ via $c(\alpha\otimes s) = c(\alpha)s$.

Definition 4.1. The twisted Dirac operator associated to (E, ∇^E) is the operator,

$$\not{\nabla} := c \circ \nabla^E : C^\infty(X,E) \to C^\infty(X,E). \tag{4.1}$$

The equations we wish to consider are (sometimes we omit the mapping c and the dimension 4 for the convenience of computations),

$$\begin{aligned}
&\not{\nabla}\psi = (d + d^*)\psi = 0,\\
&c(W^+_g) = \frac{1}{4}\langle e_i\cdot e_j\psi,\psi\rangle e^i\wedge e^j,\\
\mathit{or}, \quad &c\Big((W^+_g)_{ijkl}\Big)e^i\wedge e^j = \frac{1}{4}\langle e_k\cdot e_l\psi,\psi\rangle.
\end{aligned} \tag{4.2}$$

5. The space of conformal structures

As always, we consider a closed oriented smooth 4-manifold X. A smooth Riemannian metric g on X is a smooth section of the bundle Sym^2T^*X of symmetric 2-tensors which is positive definite everywhere. The space $\mathcal{M}$ of all Riemannian metrics on X is a convex open cone in $\Gamma(Sym^2T^*X)$. Therefore the tangent space $T_g\mathcal{M}$ is canonically isomorphic to $\Gamma(Sym^2T^*X)$. We denote by $\mathcal{D}$ the group of orientation-preserving diffeomorphisms of X is an infinite-dimensional Lie group acting on $\mathcal{M}$ by pullback. The following theorem is due to Bourguignon [cf.6]:

Theorem 5.1. *Let X be a closed oriented smooth manifold of dimension ≥ 2. The space $\mathcal{M}/\mathcal{D}$ of Riemannian structures is a stratified Hausdorff space with dense open stratum $\mathcal{M}^*/\mathcal{D}$ where I_g is the stabilizer of $\mathcal{D}$ of a metric $g \in \mathcal{M}$, also $\mathcal{M}^* = \{g \in \mathcal{M} : I_g = \{e\}\}$. One can further show that $\mathcal{M}^*/\mathcal{D}$ is a manifold.*

One considers the action of $\mathcal{D}$ on $X \times \mathcal{M}$. The quotient space descends naturally to $\mathcal{M}/\mathcal{D}$, the fibre over $[g]$ is diffeomorphic to X/I_g. When one restricts the above to $\mathcal{M}^*$, one obtains a smooth fibre bundle P over $\mathcal{M}^*/\mathcal{D}$. By definition, the fibres P_g are isometric to (X, g). One therefore concludes $P \to \mathcal{M}^*/\mathcal{D}$ is a universal family of metrics with no isometries.

Let us denote by C^∞_+ the space of positive smooth functions, then the group $\mathcal{C} := \mathcal{D} \times C^\infty_+$ acts on the quotient of the space of metrics in the following way:

$$(\phi_1, f_1) \cdot (\phi_2, f_2) = (\phi_1 \circ \phi_2, f_2 \cdot (f_1 \circ \phi_2)), \tag{5.1}$$

also $\mathcal{C}$ acts smoothly on $\mathcal{M}$ on the right:

$$\begin{aligned} \mathcal{C} \times \mathcal{M} &\to \mathcal{M} \\ ((\phi, f), g) &\mapsto f \cdot \phi^*(g). \end{aligned} \tag{5.2}$$

The stabilizer C_g of g is called the conformal isometry group of g. One can show C_g is compact unless $X \cong S^n$.

The following theorem is well-known:

Theorem 5.2. *Let X be a closed oriented smooth manifold of dimension ≥ 3. We further assume X is not diffeomorphic to S^n. The space $\mathcal{M}/\mathcal{C}$ of conformal structures is a stratified Hausdorff space with dense open stratum the manifold $\mathcal{M}^{**}/\mathcal{C}$, where $\mathcal{M}^{**} = \{g \in \mathcal{M} : C_g = \{e\}\}$. The singularities of $\mathcal{M}/\mathcal{C}$ are quotients by compact groups.*

So, we get a universal family of conformal structures $\mathfrak{C}$ over $\mathcal{M}^{**}/\mathcal{C}$.

6. The construction of the Elliptic complex

To get an elliptic complex corresponding to the data of the Gravitational monopole equations, we modify the data a little bit. So, we assume, A is a spin connection on the smooth 4-manifold X. ψ is a section of $\mathbb{S}^+$, so we relace the Dirac data $\psi \in \ker(d + d^*)$ to $\psi \in \ker D_A$. In this way, the Dirac operator becomes dependent on A and we get a more general context. Let $\widehat{A}$ be the induced connection on the line bundle $L \to X$. We denote by $F_{\widehat{A}}$ the curvature corresponding to the connection $\widehat{A}$. We put no restriction on $F_{\widehat{A}}$. All the restrictions are on the self-dual part of the Weyl tensor of X.

In the case of a line bundle, the gauge group $\mathcal{G} = \mathcal{M}(X, U(1))$ as a space of maps is well-defined as it is only dependent on the transition functions. The action of $\mathcal{G}$ on the pair (A, ψ) is given in the following way:

$$\lambda : (A, \psi) \to (A - \lambda^{-1} d\lambda, \lambda\psi), \tag{6.1}$$

and on $\widehat{A}$ by

$$\widehat{A} - 2\mathbf{i}\lambda^{-1} d\lambda. \tag{6.2}$$

We verify the following:

$$D_{A - \lambda^{-1} d\lambda}(\lambda\psi) = \lambda D_A \psi + d\lambda \cdot \psi - d\lambda \cdot \psi. \tag{6.3}$$

The self-dual part $F^+_{\widehat{A}}$ of the curvature tensor $F_{\widehat{A}}$ also remains invariant

$$F^+_{\widehat{A}-2\lambda^{-1}d\lambda} = F^+_{\widehat{A}} - 2d^+(\lambda^{-1}d\lambda) = F^+_{\widehat{A}}. \tag{6.4}$$

Since the metric $g_{ij} \mapsto g_{AA'BB'} = \varepsilon_{AB}\varepsilon_{A'B'}$, corresponding $U(1)$-action on the metric is just multiplication by $|\lambda|^2 = 1$, therefore W^+ remains invariant, on the other hand

$$\langle e_i e_j \lambda\psi, \lambda\psi\rangle = |\lambda|^2\langle e_i e_j \psi, \psi\rangle = \langle e_i e_j \psi, \psi\rangle, \tag{6.5}$$

as $|\lambda| = 1$.

Now we study the kernel of the linearized operator $T_g W^+ : T_g\mathcal{M} \to \Gamma(Sym^2_0\wedge^2_+)$. The deformations we shall consider will be represented by the first cohomology of the complex

$$\Gamma(TX) \oplus \Gamma(\mathbb{R}) \xrightarrow{\tau^*} \Gamma(Sym^2_0\wedge^2_+) \xrightarrow{TW^+} \Gamma(Sym^2_0\wedge^2_+), \tag{6.6}$$

where $\mathbb{R}$ represents trivial $\mathbb{R}$-bundle over X. $\Gamma(Sym^2T^*X) = T_g\mathcal{M} = Hom(\wedge^2_+, \wedge^2_-)\oplus\mathbb{R}$.. But $Hom(\wedge^2_+, \wedge^2_-) \cong \wedge^2_+ \otimes \wedge^2_-$. If we want to mod out the trivial line bundle $\mathbb{R}$ then we must work with $\mathcal{M}/C^\infty_+$, and the section space $\Gamma(\mathbb{R})$ is replaced by the orbit space $C^\infty_+(g)$.

The problem is not underdetermined as the Gravitational monopole equation (4.2) gives additional restriction on the scalar curvature, that is $s = s_g < 0$, hence a linearization at g gives the following complex

$$\Gamma(TX) \xrightarrow{\delta^*} \Gamma(Sym^2T^*X) \xrightarrow{TW_+\oplus T_g} \Gamma(Sym^2_0\wedge^2_+) \oplus \Gamma(\mathbb{R}). \tag{6.7}$$

Theorem 6.1. *The complex (6.7) is elliptic with index equals to*

$$\frac{1}{2}(29|\sigma(X)| - 15\chi(X)), \tag{6.8}$$

where $\chi(X)$ is the Euler characteristic of X and $\sigma(X)$ are the signature of X.

By the index theorem for the twisted Dirac operator

$$Index(D_A) = -\int_X ch(\sqrt{L})\widehat{A}(X). \tag{6.9}$$

The Chern character is known to be

$$Ch(\sqrt{L}) = 2(1 + c_1(L)^2 + \cdots) \tag{6.10}$$

the $\widehat{A}$-genus is

$$\widehat{A}(X) = 1 - \frac{1}{24}p_1(X) + \cdots, \tag{6.11}$$

where $p_1(X)$ is the first-Pontrjagin class. The top degree form of $Ch(\sqrt{L})\widehat{A}(X)$ is $\frac{1}{12}p_1(X) + c_1(L)^2$, but $\frac{1}{3}p_1(X) = \sigma$, therefore we have the result

$$Index(D_A) = c_1(\sqrt{L})^2 - \frac{\sigma}{4}. \tag{6.12}$$

Therefore we have the following theorem

Theorem 6.2. *The moduli space of solutions of the Gravitational monopole equation has dimension (assuming the index $\sigma(X) \geq 0$)*

$$\begin{aligned} &\frac{1}{2}(29\sigma(X) - 15\chi(X)) + c_1(\sqrt{L})^2 - \frac{\sigma}{4} \\ &= c_1(\sqrt{L})^2 + \frac{57\sigma(X) - 30\chi(X)}{4} \end{aligned} \tag{6.13}$$

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