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HYPERLOGARITHMIC FUNCTIONAL EQUATIONS ON DEL PEZZO SURFACES

ANA-MARIA CASTRAVET & LUC PIRIO

ABSTRACT. For any $d \in \{1, \dots, 6\}$, we prove that the web of conics on a del Pezzo surface of degree d carries a functional identity whose components are antisymmetric hyperlogarithms of weight $7 - d$. Our approach is uniform with respect to d and relies on classical results about the action of the Weyl group on the set of lines on the del Pezzo surface. These hyperlogarithmic functional identities are natural generalizations of the classical 3-term and (Abel's) 5-term identities satisfied by the logarithm and the dilogarithm, which correspond to the cases when $d = 6$ and $d = 5$ respectively.

1. INTRODUCTION

1.1. Functional equations of polylogarithms. The classical logarithm Log satisfies *Cauchy's identity*

$$(1) \quad \text{Log}(x) + \text{Log}(y) - \text{Log}(xy) = 0$$

for all $x, y > 0$, and this functional identity is fundamental in mathematics.

Several authors of the XIXth century have independently discovered equivalent forms of the following identity

$$(\mathcal{A}b) \quad R(x) - R(y) - R\left(\frac{x}{y}\right) - R\left(\frac{1-y}{1-x}\right) + R\left(\frac{x(1-y)}{y(1-x)}\right) = 0,$$

satisfied for all x, y such that $0 < x < y < 1$, where R stands for *Rogers' dilogarithm*, defined by

$$(2) \quad R(x) = \text{Li}_2(x) + \frac{1}{2} \text{Log}(x) \text{Log}(1-x) - \frac{\pi^2}{6}$$

for $x \in (0, 1)$, where Li_2 stands for the classical bilogarithm, the weight 2 polylogarithm.

The identity $(\mathcal{A}b)$ is nowadays called *Abel's identity* of the dilogarithm, hence the notation. It can be seen as a weight 2 generalization of Cauchy's identity (1). To see in which way, recall the *weight $n \geq 1$ polylogarithm* Li_n , classically defined on the unit disk $\mathbf{D} = \{z \in \mathbf{C}, |z| < 1\}$ as the sum of the convergent series

$$\text{Li}_n(z) = \sum_{k \geq 1} z^k / k^n.$$

The first polylogarithm is related to the usual logarithm through the relation $\text{Li}_1(z) = -\text{Log}(1-z)$ for $z \in \mathbf{D}$, and both Log and R can be considered as suitable versions of the first two polylogarithms such that the two functional identities (1) and $(\mathcal{A}b)$ hold true.

Polylogarithms are special functions of great interest which satisfy properties generalizing those of the logarithm and the dilogarithm. In particular, they satisfy functional equations of the form

$$(3) \quad \sum_{i=1}^M c_i \text{Li}_n(U_i) = \text{Li}_{n-1}$$

where the c_i 's are rational coefficients, the U_i 's are multi-variable rational functions and with L_{n-1} a rational expression in polylogarithmic functions of weight at most $n - 1$.

Since the early XIX-th century (works of Spence, Kummer, Abel, etc) to nowadays, many authors have discovered functional identities of the above form satisfied by (some version of) polylogarithms \mathbf{Li}_n of weight $n \leq 7$. Multi-variable generalizations of polylogarithms have been considered as well, in particular their functional equations. The subject is currently very active.¹ Polylogarithms are connected to several distinct fields in mathematics such as hyperbolic geometry (volumes of hyperbolic polytopes), K -theory of number fields (Zagier's conjecture), theory of periods and multizeta values, scattering amplitudes in higher energy physics, theory of cluster algebras, etc.² In particular, it is now clearly established that knowing functional identities of the form (3) is important, cf. [G1, §2]. However, in spite of the important number of recent works on the subject, the functional identities satisfied by polylogarithms are still not well understood.

Identities of the form (3) are known to exist only for $n \leq 7$ and for the higher weights ($n = 6, 7$) were obtained by computer aided calculations (see [Ga]). The general belief is that, for any $n \geq 1$, there should exist a fundamental identity of the form (3) satisfied by \mathbf{Li}_n from which any other could be formally obtained (for instance, see the last paragraph of [Gri, §4.1]). The first interesting case to be considered is for weight 2, for which Abel's identity is the evoked fundamental one, a result which has been proved only recently in [dJ]. For weight 3, the 22-term trilogarithmic equation in three variables found by Goncharov in [G3] may be the fundamental one but, as far we are aware of, there is no proof until now. The weight 4 case is the subject of the recent work [GR] by Goncharov and Rudenko. Using the cluster structure of the moduli spaces $\mathcal{M}_{0,n+3}$, they construct a functional identity for the tetralogarithm which allows them to prove Zagier's conjecture in weight 4. This identity is expected to play the same role for the tetralogarithm as the one played by Abel's identity for the dilogarithm.

1.2. Hyperlogarithms. Hyperlogarithms are generalization of polylogarithms and they go back to Poincaré. They are multivalued holomorphic functions on \mathbf{P}^1 which can be obtained by iterated integrations of some given rational 1-forms with logarithmic singularities on the Riemann sphere. More precisely, let $\sigma_1, \dots, \sigma_{m+1}$ be $m+1$ pairwise distinct points of \mathbf{P}^1 . We fix an affine coordinate z such that $\sigma_{m+1} = \infty$. Then the 1-forms $\omega_s = dz/(z - \sigma_s)$ for $s = 1, \dots, m$ form a basis of the space of global logarithmic 1-forms on \mathbf{P}^1 with poles in $\Sigma = \{\sigma_s\}_{s=1}^{m+1}$. We set $Z = \mathbf{P}^1 \setminus \Sigma$.

Given any tuple $(s_k)_{k=1}^w$ of elements in $\{1, \dots, m\}$, the *weight w hyperlogarithm* $L_{\omega_{s_1} \dots \omega_{s_w}}$ is the multivalued function on Z defined inductively as follows:

$$L_{\omega_{s_w}}(z) = \int^z \omega_{s_w} = \text{Log}(z - \sigma_{s_w}) \quad \text{and} \quad L_{\omega_{s_1} \dots \omega_{s_w}}(z) = \int^z \frac{L_{\omega_{s_2} \dots \omega_{s_w}}(u)}{u - \sigma_{s_1}} du$$

The polylogarithmic functions (such as Log, Rogers' dilogarithm R , or all the classical polylogarithm \mathbf{Li}_n) are particular instances of hyperlogarithms in the specific case when $\Sigma = \{0, 1, \infty\}$. If the properties of polylogarithms, in particular the functional equations they satisfy, have been studied intensely, this is much less the case for more general hyperlogarithms (however see [We] or the more recent [Br]). Several recent works have shown that hyperlogarithms are relevant for computing certain scattering amplitudes in higher energy physics (see for instance the PhD thesis [Pa] or the recent "white paper" [B&al], especially the fifth section therein).

¹E.g., see the works Goncharov, Gangl, Goncharov-Rudenko, Charlton-Gangl-Radchenko, Rudenko.

²For more details, we refer to the surveys [Za] or [G2].

In this paper, we describe generalizations in weight 3, 4, 5 and 6 of the 3-term and 5-term identities of the logarithm and dilogarithm respectively. These identities are similar to the two latter classical identities, but involve non polylogarithmic hyperlogarithms. For this purpose, we introduce a geometric viewpoint on Abel's identity (\mathcal{Ab}) by relating it to the conic fibrations of a quintic del Pezzo surface. The generalization will involve the conic fibrations of del Pezzo surfaces of degree ≤ 6 .³

1.3. Abel's identity on the quintic del Pezzo surface. The five rational arguments

$$U_1 = x, \quad U_2 = y, \quad U_3 = \frac{x}{y}, \quad U_4 = \frac{1-y}{1-x} \quad \text{and} \quad U_5 = \frac{x(1-y)}{y(1-x)}$$

of R in (\mathcal{Ab}) can be interpreted geometrically as follows: let

$$\beta : X_4 = \mathbf{Bl}_{p_1, \dots, p_4}(\mathbf{P}^2) \longrightarrow \mathbf{P}^2$$

be the blow-up of the complex projective plane at the 4 points in general position $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$ and $p_4 = [1 : 1 : 1]$. The surface is the *quintic del Pezzo surface*. It carries five fibrations in conics $\phi_i : X_4 \rightarrow \mathbf{P}^1$ ($i = 1, \dots, 5$) which coincide with the compositions $U_i \circ \beta : X_4 \dashrightarrow \mathbf{P}^1$ as rational functions. It follows that Abel's identity can be written

$$(\mathcal{Ab}_{X_4}) \quad \sum_{i=1}^5 \epsilon_i R(\phi_i) = 0$$

for some constants $\epsilon_1, \dots, \epsilon_5$ equal to 1 or -1 , this identity holding true locally at any sufficiently general point of X_4 for suitable branches of Rogers dilogarithm.

1.4. Main result: generalization to del Pezzo surfaces of degree ≤ 6 . Let $3 \leq r \leq 8$ and let

$$X_r = \mathbf{Bl}_{p_1, \dots, p_r}(\mathbf{P}^2)$$

be the blow-up of the projective plane at r points in general position. Then X_r is a del Pezzo surface of degree $9 - r$, i.e., the anti-canonical class $-K_{X_r}$ is ample and with self-intersection $(-K_{X_r})^2 = 9 - r$. If $3 \leq r \leq 6$, then the complete linear system $-K_{X_r}$ defines an embedding $X_r \hookrightarrow \mathbf{P}^{9-r}$ such that the degree of X_r is $9 - r$. We define the *degree* of a curve $C \subset X_r$ to be $C \cdot (-K_{X_r})$. Smooth rational curves in X_r of degree 1, respectively 2, are called *lines*, respectively *smooth conics*. A conic fibration on X_r is the equivalence class (up to post composition with an element of PGL_2) of a morphism $X_r \rightarrow \mathbf{P}^1$ such that a general fiber is a smooth conic.

The following facts are well known:

- (i). The number l_r of lines in X_r is finite;
- (ii). The number κ_r of conic fibrations on X_r is finite as well;
- (iii). Any conic fibration $X_r \rightarrow \mathbf{P}^1$ has exactly $r - 1$ reducible fibers, each a union of two lines in X_r intersecting transversely at one point.
- (iv). The Picard group $\mathrm{Pic}(X_r)$ is free and is acted upon by a certain Weyl group W_r . Moreover, this action preserves the intersection product.

³The 3-term identity of the logarithm can also be considered from a geometric perspective, but it is less meaningful from this point of view, because it is "too simple". This is why we only consider the case of Abel' equation.

The values of l_r and κ_r for $3 \leq r \leq 8$ are given in the following table:

r	3	4	5	6	7	8
l_r	6	10	16	27	56	240
κ_r	3	5	10	27	126	2160

Let $\phi_1, \dots, \phi_{\kappa_r} : X_r \rightarrow \mathbf{P}^1$ be κ_r pairwise non equivalent conic fibrations. We denote by L_r the divisor of X_r whose the irreducible components are all the lines in X_r and we set

$$Y_r = X_r \setminus L_r.$$

From (iii), we know that the complement Σ_i of $\phi_i(Y_r)$ in \mathbf{P}^1 is a finite set with $r - 1$ elements denoted by $\sigma_i^1, \dots, \sigma_i^{r-1}$. One assumes that ϕ_i has been chosen such that one of the σ_i^t 's, say σ_i^{r-1} , coincides with $\infty \in \mathbf{P}^1$. Then the rational differentials $\omega_i^t = dz/(z - \sigma_i^t)$ for $t = 1, \dots, r - 2$ form a basis of the space of logarithmic 1-forms on \mathbf{P}^1 with poles along Σ_i .

For all $i = 1, \dots, \kappa_r$, let AI_i^{r-2} be the *complete antisymmetric hyperlogarithm* of weight $r - 2$ on $Z_i = \mathbf{P}^1 \setminus \Sigma_i$, defined as the antisymmetrization of the hyperlogarithm $L_{\omega_i^1 \dots \omega_i^{r-2}}$ with respect to the logarithmic 1-forms $\omega_i^1, \dots, \omega_i^{r-2}$, i.e.,

$$AI_i^{r-2} = \text{Asym}(L_{\omega_i^1 \dots \omega_i^{r-2}}) = \frac{1}{(r-2)!} \sum_{\nu \in \mathfrak{S}_{r-2}} (-1)^\nu L_{\omega_i^{\nu(1)} \dots \omega_i^{\nu(r-2)}}$$

where, for any $\nu \in \mathfrak{S}_{r-2}$, we denote by $(-1)^\nu$ the signature of ν . Each AI_i^{r-2} is uniquely defined up to sign. Our main result is the following:

Theorem 1.1. *There exists $(\epsilon_i)_{i=1}^{\kappa_r} \in \{\pm 1\}^{\kappa_r}$, unique up to a global sign, such that for any $y \in Y_r$ and for a suitable choice of the branch of the hyperlogarithm AI_i^{r-2} at $y_i = \phi_i(y) \in \mathbf{P}^1$ for each $i = 1, \dots, \kappa_r$, the following functional identity holds true on an open neighbourhood of y in Y_r :*

$$\mathbf{HLog}(X_r) \quad \sum_{i=1}^{\kappa_r} \epsilon_i AI_i^{r-2}(\phi_i) = 0.$$

A few comments:

- The identity $\mathbf{HLog}(X_3)$ is nothing else but the logarithm identity (1) and $\mathbf{HLog}(X_4)$ coincides with the geometric identity $(\mathcal{A}b_{X_4})$ hence is equivalent to Abel's relation $(\mathcal{A}b)$. In contrast, the four other identities $\mathbf{HLog}(X_r)$ for $r = 5, 6, 7, 8$ are new.
- For any $i = 1, \dots, \kappa_r$, the suitable branch of the hyperlogarithm AI_i^{r-2} from the statement of the theorem is defined in a precise and constructive way (see 8). Furthermore, in Theorem 3.1 we prove an invariant algebraic version of Theorem 1.1 by constructing (an algebraic equivalent of) each term $\epsilon_i AI_i^{r-2}(\phi_i)$ by means of the natural action of the Weyl group W_r on only one term, which we may assume to be $AI_1^{r-2}(\phi_1)$.
- At least when $r \leq 7$, there is a conceptual interpretation of why $\mathbf{HLog}(X_r)$ holds true in terms of the space $\mathbf{C}^{\mathcal{L}_r}$ freely spanned by the set \mathcal{L}_r of lines contained in X_r . This space is acted upon in a natural way by W_r and from a representation-theoretic perspective, the left-hand side of $\mathbf{HLog}(X_r)$ can be interpreted as the image of the signature representation \mathbf{sign}_r of the Weyl group W_r in the $(r - 2)$ -th wedge product of $\mathbf{C}^{\mathcal{L}_r}$. The reason why $\sum_{i=1}^{\kappa_r} \epsilon_i AI_i^{r-2}(\phi_i)$ vanishes identically is that \mathbf{sign}_r does not appear with positive multiplicity in the decomposition of $\wedge^{r-2} \mathbf{C}^{\mathcal{L}_r}$ in irreducible W_r -modules.

1.5. Structure of the paper. Throughout the paper, we work in the complex analytic or algebraic setting.

In Section §2, we recall the basic facts about hyperlogarithms and del Pezzo surfaces which will be used in the rest of the paper. In particular, we explain how the functional identities satisfied by hyperlogarithms can be proved algebraically (*cf.* Proposition 2.2). Section §3 is the main section and that is where Theorem 1.1 is proved. Using Proposition 2.2, its proof is essentially reduced to the verification that a certain (antisymmetric) tensorial identity $\mathbf{hlog} = 0$ holds true. We include at the end of Section §3 some considerations regarding possible generalizations to higher dimensions (blow-ups of projective spaces at general points). Finally in Section §4, we make the identity $\mathbf{HLog}(X_5)$ explicit in some affine coordinates.

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2. Preliminaries

In this section, we recall some properties of hyperlogarithms and del Pezzo surfaces.

2.1. Hyperlogarithms. Hyperlogarithms are multivalued holomorphic functions on \mathbf{P}^1 which were used by Poincaré and Lappo-Danilevsky for building solutions to linear differential equations with regular singular points on the Riemann sphere. As modern references about hyperlogarithms, the reader can consult [We, Br] or [BPP, §2.3].

2.1.1. Let $n \geq 1$ and $\sigma_1, \dots, \sigma_m$ be n pairwise distinct complex numbers. We set

$$\Sigma = \{\sigma_1, \dots, \sigma_m, \infty\} \subset \mathbf{P}^1 \quad \text{and} \quad Y = \mathbf{P}^1 \setminus \Sigma.$$

The 1-forms $\omega_k = d \operatorname{Log}(z - \sigma_k) = dz/(z - \sigma_k)$ for $k = 1, \dots, m$ form a basis of the space

$$\mathcal{H}_\Sigma = \mathbf{H}^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^1(\operatorname{Log} \Sigma))$$

of global rational 1-forms on \mathbf{P}^1 with logarithmic poles along Σ .

We fix a base point $y \in Y$. For any word $\omega_{k_1} \omega_{k_2} \cdots \omega_{k_w}$ on the ω_k 's, of length $w \geq 1$, we define the *hyperlogarithm associated to it at y* as the holomorphic germ at this point, denoted by $L_{\omega_{k_1} \dots \omega_{k_w}}^y$ defined inductively on the length w by successive integrations performed on a sufficiently small neighborhood of y , according to the following relations:

$$L_{\omega_{k_w}}^y(z) = \int_y^z \omega_{k_w} = \operatorname{Log} \left(\frac{z - \sigma_{k_w}}{y - \sigma_{k_w}} \right) \quad \text{and} \quad L_{\omega_{k_1} \dots \omega_{k_w}}^y(z) = \int_y^z \frac{L_{\omega_{k_2} \dots \omega_{k_w}}^y(u)}{u - \sigma_{k_1}} du \quad \text{for } w > 1,$$

for any z sufficiently close to y on \mathbf{P}^1 . The germ $L_{\omega_{k_1} \dots \omega_{k_w}} \in \mathcal{O}_{Y,y}$ admits analytic continuation along any continuous path $\gamma_z : [0, 1] \rightarrow \mathbf{P}^1 \setminus \Sigma$ joining y to an arbitrary point $z \in Y$. The value

at z of this analytic continuation only depends on the homotopy class of γ_z and is easily seen to coincide with the iterated integral of the tensor $\omega_{k_1} \otimes \cdots \otimes \omega_{k_w} \in (\mathcal{H}_\Sigma)^{\otimes w}$ along γ_z : one has

$$L_{\omega_{k_1} \dots \omega_{k_w}}^y(z) = \int_{\gamma_z} \frac{du}{u - \sigma_{k_1}} \otimes \frac{du}{u - \sigma_{k_2}} \otimes \cdots \otimes \frac{du}{u - \sigma_{k_w}}.$$

The germ $L_{\omega_{k_1} \dots \omega_{k_w}}^y$ gives rise to a global but multivalued holomorphic function on Y , with branch points at the σ_k 's, which we will still refer to as the hyperlogarithm associated to $\omega_{k_1} \dots \omega_{k_w}$ and we denote by $L_{\omega_{k_1} \dots \omega_{k_w}}$.

More formally, we consider the map (where \mathbb{I} stands for ‘‘Iterated Integral’’)

$$(4) \quad \mathbb{I}_Y^y : \oplus_{w \geq 0} (\mathcal{H}_\Sigma)^{\otimes w} \longrightarrow \mathcal{O}_{\mathbf{P}^1, y}, \quad \omega_{k_1} \otimes \omega_{k_2} \otimes \cdots \otimes \omega_{k_w} \longmapsto L_{\omega_{k_1} \dots \omega_{k_w}}^y.$$

which in addition to being \mathbf{C} -linear, can be proved to be a morphism of algebras if $\oplus_{w \geq 0} (\mathcal{H}_\Sigma)^{\otimes w}$ is endowed with the so-called ‘‘shuffle product’’ (but we will not use this property in the rest of the paper). The image $\text{Im}(\mathbb{I}_Y^y)$ is a complex subalgebra of $\mathcal{O}_{\mathbf{P}^1, y}$ and its elements are called (*germs at y of*) *hyperlogarithms*. Moreover, the morphism (4) is injective. Consequently, for any germ of hyperlogarithm $L \in \mathcal{O}_{\mathbf{P}^1, y}$, the minimum $w(L)$ of integers $w \geq 0$ such that L belongs to the image of $\oplus_{w' \geq w} (\mathcal{H}_\Sigma)^{\otimes w'}$ by \mathbb{I}_Y^y is well-defined and is called the *weight* of L .

As multivalued functions on \mathbf{P}^1 , the monodromy of hyperlogarithms can be proved to be unipotent (see [We, Thm. 8.2]) from which it follows that these functions also form an algebra and that the notion of weight still makes sense for them.

2.1.2. The most classical example is for $m = 2$ with $\sigma_1 = 0$ and $\sigma_2 = 1$ which encompasses the case of classical polylogarithms. Indeed, setting $\eta_0 = dz/z$ and $\eta_1 = dz/(1 - z)$ in this special case, as multivalued hyperlogarithms on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, one has

$$\text{Log} = L_{\eta_0}, \quad R = \frac{1}{2} (L_{\eta_0 \eta_1} - L_{\eta_1 \eta_0}) \quad \text{and} \quad \text{Li}_{n+1} = L_{\eta_0^n \eta_1} \quad \text{for any } n \geq 0.$$

Working locally with germs of hyperlogarithms is more involved but removes all ambiguity regarding the choice of a branch of the functions considered. For instance, for any $y \in \mathbf{P}^1 \setminus \{0, 1, \infty\}$, one has that the weight 2 hyperlogarithm at y whose symbol is $\frac{1}{2}(\eta_0 \eta_1 - \eta_1 \eta_0)$ is the holomorphic function defined by

$$R^y(z) = \frac{1}{2} (L_{\eta_0 \eta_1}(z) - L_{\eta_1 \eta_0}(z)) = \frac{1}{2} \int_y^z \left(\frac{\text{Log}\left(\frac{u-1}{y-1}\right)}{u} - \frac{\text{Log}\left(\frac{u}{y}\right)}{u-1} \right) du$$

for any $z \in (\mathbf{P}^1, y)$. This hyperlogarithm has to be seen as a holomorphic version, localized at y , of Rogers’ dilogarithm defined in 2.

2.1.3. We now define the hyperlogarithms involved in this paper, noted by AI^w . Even if our results is for weights w less than or equal to 6, the definition of AI^w is completely uniform in w , hence we will not impose any restriction on the weight in this subsection.

We use the notation of §2.1.1 again: $\Sigma = \{\sigma_1, \dots, \sigma_m, \infty\}$, $\omega_k = dz/(z - \sigma_k)$ for $k = 1, \dots, m$, etc. We introduce a special class of hyperlogarithms on \mathbf{P}^1 , with respect to Σ , of weight $m = |\Sigma| - 1$.

For a \mathbf{C} -vector space V , we identify $\wedge^m V$ with its image in $V^{\otimes m}$ under the standard embedding:

$$\wedge^m V \hookrightarrow V^{\otimes m}, \quad v_1 \wedge \dots \wedge v_m \mapsto \frac{1}{m!} \left(\sum_{\tau \in \mathfrak{S}_m} (-1)^\tau v_{\tau(1)} \otimes \dots \otimes v_{\tau(m)} \right),$$

where $(-1)^\tau$ stands for the signature of τ for any permutation $\tau \in \mathfrak{S}_m$.

Definition 2.1. *The (complete) anti-symmetric hyperlogarithm AI^m of weight m on \mathbf{P}^1 , with respect to Σ , is the hyperlogarithm whose germ at any $y \in Y = \mathbf{P}^1 \setminus \Sigma$ is obtained by taking the image of $\omega_1 \wedge \dots \wedge \omega_m \in \wedge^m \mathcal{H}_\Sigma \subset (\mathcal{H}_\Sigma)^{\otimes m}$ under the map (4): as germs at y , one has*

$$AI_\Sigma^m = \Pi_Y^y(\omega_1 \wedge \dots \wedge \omega_m).$$

One verifies that $\omega_1 \wedge \dots \wedge \omega_m \in \wedge^m \mathcal{H}_\Sigma$ is canonically defined, up to a sign. It follows that $\pm AI_\Sigma^m$ is canonically defined by Σ . Here are some easy remarks about the first three examples:

- $m = 1$ and $\sigma_1 = 0$; one has $AI_{\{0, \infty\}}^1 = \text{Log}$ up to sign;
- $m = 2$ and $\sigma_1 = 0, \sigma^2 = 1$; up to sign, one recovers the holomorphic version of Rogers' dilogarithm discussed above since $AI_{\{0, 1, \infty\}}^2 = R^y$ as germs at any $y \in \mathbf{P}^1 \setminus \{0, 1, \infty\}$;
- the case $m = 3$ is new since, the weight 3 antisymmetric hyperlogarithm has not been considered in the literature before as far we know. For any $y \in \mathbf{P}^1 \setminus \Sigma$ with $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \infty\}$, one can give an explicit integral expression for AI_Σ^3 (see (19)). However, one can prove that AI_Σ^3 can be expressed as the following linear combination of products of antisymmetric polylogarithms of weight 1 or 2 since for suitable choices of the sign of AI_Σ^3 and of the weight two hyperlogarithms $AI_{\Sigma \setminus \{\sigma_k\}}^2$'s for $k = 1, 2, 3$, the following relation holds true

$$AI_\Sigma^3(z) = \frac{1}{3} \sum_{k=1}^3 (-1)^{k-1} \text{Log} \left(\frac{z - \sigma_k}{y - \sigma_k} \right) \cdot AI_{\Sigma \setminus \{\sigma_k\}}^2(z)$$

for any $z \in \mathbf{P}^1$ sufficiently close of the previously fixed base point y .

2.1.4. Pull-backs of hyperlogarithms. Let Y be a (not necessarily compact) complex manifold and let $\mathcal{H} \subset \mathbf{H}^0(Y, \Omega_Y^1)$ be a subspace of holomorphic 1-forms on Y , such that:

$$(5) \quad \text{For all } \omega, \omega' \in \mathcal{H}, \text{ one has } d\omega = 0 \text{ and } \omega \wedge \omega' = 0.$$

The conditions (5) are satisfied if, for example, $\mathcal{H} = \phi^* \mathbf{H}^0(C, \Omega_C^1)$, for some regular submersion $\phi : Y \rightarrow C$, with C a smooth (not necessarily compact) curve. The conditions (5) ensure that for any holomorphic 1-forms $\omega_1, \dots, \omega_w$ in \mathcal{H} , the iterated integral $\Pi_{\omega_1 \omega_2 \dots \omega_w}^y = \int^\bullet \omega_1 \otimes \omega_2 \otimes \dots \otimes \omega_w$, defined inductively as in §2.1.1, depends only on the homotopy class of the path γ_z . Hence, for all $m \geq 1$, there is \mathbf{C} -linear map $\Pi_Y^y : \oplus_{w \geq 0} \mathcal{H}^{\otimes w} \rightarrow \mathcal{O}_{Y, y}$, defined as in (4). Furthermore, this map is an injective morphism of complex algebras.

A special situation occurs when the conditions (5) are not necessarily satisfied for all elements of \mathcal{H} , but there exist subspaces $\mathcal{H}_i \subset \mathcal{H}$ for $i = 1, \dots, d$, such that for each i , \mathcal{H}_i satisfies (5). In this case, we have again a well-defined *injective* \mathbf{C} -linear map given by the iterated integrals on the subspaces $\sum_{i=1}^d (\mathcal{H}_i)^{\otimes w} \subset \mathcal{H}^{\otimes w}$, for all $w \geq 0$:

$$(6) \quad \Pi_Y^y : \oplus_{w \geq 0} \left(\sum_{i=1}^d \mathcal{H}_i^{\otimes w} \right) \rightarrow \mathcal{O}_{Y, y}.$$

2.1.5. Hyperlogaritms for webs. Fix $m \geq 1$. The situation we consider here is when X a complex projective manifold, $\phi_i : X \rightarrow \mathbf{P}^1$ surjective morphisms (with $i = 1, \dots, d$), such that there exists subsets of $m+1$ distinct points $\Sigma_i = \{\sigma_i^1, \dots, \sigma_i^m, \infty\} \subset \mathbf{P}^1$, such that $\phi_i : X \setminus \phi_i^{-1}(\Sigma_i) \rightarrow \mathbf{P}^1 \setminus \Sigma_i$ is a regular submersion for all i , and the union $D \subset X$ of all divisors $\phi_i^{-1}(\sigma_i^k) \subset X$, for all i and $k = 1, \dots, m$ is such that one has $d\phi_i \wedge d\phi_j \neq 0$ on $Y = X \setminus D$, for all $i, j = 1, \dots, d$ distinct. Then the maps ϕ_i ($i = 1, \dots, d$) define a regular d -web of hypersurfaces on Y .

In such a situation, we consider the following notations:

- we denote by $\mathcal{H} = \mathbf{H}^0(X, \Omega_X^1(\text{Log} D)) \subset \mathbf{H}^0(Y, \Omega_Y^1)$ the space of logarithmic 1-forms on X with logarithmic poles along to D ;
- for $i = 1, \dots, d$, we set $Y_i = \mathbf{P}^1 \setminus \Sigma_i$ and

$$- \theta_i^j = dz/(z - \sigma_i^k) \text{ for } k = 1, \dots, m, \text{ which form a basis of}$$

$$\mathcal{H}_{\Sigma_i} = \mathbf{H}^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^1(\text{Log } \Sigma_i))$$

$$- \Theta_i^j = \phi_i^*(\theta_i^j) = d\phi_i/(\phi_i - \sigma_i^k) \text{ for } k = 1, \dots, m, \text{ which form a basis of}$$

$$\mathcal{H}_i = \phi_i^*(\mathcal{H}_{\Sigma_i}) \subset \mathcal{H};$$

$$- \theta_i = \theta_i^1 \wedge \dots \wedge \theta_i^m \in \wedge^m \mathcal{H}_{\Sigma_i} \subset (\mathcal{H}_{\Sigma_i})^{\otimes m} \text{ and } \Theta_i = \phi_i^*(\theta_i) = \wedge_{k=1}^m \Theta_i^k \in \wedge^m \mathcal{H}_i \subset (\mathcal{H}_i)^{\otimes m};$$

- for any $y \in Y$, we set $y_i = \phi_i(y) \in \mathbf{P}^1 \setminus \Sigma_i$ and we consider the (germs of) weight m hyperlogarithms

$$AI_{\Sigma_i}^m = \Pi_{Y_i}^{y_i}(\theta_i) \in \mathcal{O}_{\mathbf{P}^1, y_i} \quad \text{for } i = 1, \dots, d \quad \text{and} \quad AI_i^m = \Pi_Y^y(\Theta_i) \in \mathcal{O}_{Y, y}.$$

One verifies easily that for any i , the following relation holds true as germs on Y at y :

$$AI_i^m = AI_{\Sigma_i}^m \circ \phi_i.$$

For any $i = 1, \dots, d$, the hyperlogarithm AI_i^m (or equivalently $AI_{\Sigma_i}^m$) is only well-defined up to multiplication by -1 . For each i , we fix one of the two possible choices for AI_i^m . The following result, although elementary to prove, is key since it will allow us to handle algebraically the functional identity we want to establish in §3:

Proposition 2.2. *For $c_1, \dots, c_d \in \mathbf{C}$, the following statements are equivalent:*

- One has $\sum_{i=1}^d c_i \Theta_i = 0$ in $\wedge^n \mathcal{H} \subset \mathcal{H}^{\otimes n}$.
- There exists $y \in Y$ such that $\sum_{i=1}^d c_i AI_{\Sigma_i}^m(\phi_i) = 0$ as a holomorphic germ at y on Y .
- For any $y \in Y$, $\sum_{i=1}^d c_i AI_{\Sigma_i}^m(\phi_i) = 0$ as a holomorphic germ at y on Y .
- One has $\sum_{i=1}^d c_i AI_{\Sigma_i}^m(\phi_i) = 0$ as multivalued functions on Y .

Proof. For a point $y \in Y$, we have $\sum_{i=1}^d c_i AI_{\Sigma_i}^m(\phi_i) = \Pi_Y^y(\sum_{i=1}^d c_i \Theta_i)$, where Π_Y^y is the integration map in (6). The statement now follows from the fact that this map is injective. \square

2.2. Del Pezzo surfaces. Del Pezzo surfaces are smooth projective surfaces with ample anti-canonical line bundle. A del Pezzo surface is isomorphic to either $\mathbf{P}^1 \times \mathbf{P}^1$ or a blow-up $\mathbf{Bl}_{p_1, \dots, p_r}(\mathbf{P}^2)$ ($r \geq 8$) at r points p_1, \dots, p_r in general position in \mathbf{P}^2 .

In what follows we consider del Pezzo surfaces $X_r = \mathbf{Bl}_{p_1, \dots, p_r}(\mathbf{P}^2)$ for $3 \leq r \leq 8$. We fix a blow-up map $\beta = \beta_r : X_r \rightarrow \mathbf{P}^2$. We refer to [Ma, Chap.IV] or [Do1, Chap.8] for general facts about del Pezzo surfaces. Here we make a list of the properties that we will use.

- (1). The Picard group $\text{Pic}(X_r)$ is a free abelian group generated by the classes e_i of the exceptional divisors $\beta^{-1}(p_i)$ (for $i = 1, \dots, r$) and the class h of the preimage under β of a general line in \mathbf{P}^2 . The intersection pairing on X_r is determined by $h^2 = 1$, $h \cdot e_i = 0$, $e_i \cdot e_j = -\delta_{ij}$, for all $i, j \in \{1, \dots, r\}$.
- (2). The canonical divisor is $K = K_{X_r} = -3h + \sum_{i=1}^r e_i$ and the *degree* of X_r is $(-K)^2 = 9 - r$.
- (3). A *line* on X_r is a smooth curve $\ell \subset X_r$ with $K \cdot \ell = \ell^2 = -1$. Such a line is necessarily a smooth rational curve and it can be naturally identified with its class in $\text{Pic}(X_r)$. We denote by \mathcal{L}_r the set of lines on X_r .
- (4). A *conic* on X_r is a curve $C \subset X_r$ with $C \cdot K = -2$ and $C^2 = 0$. When C is smooth, it is necessarily a smooth rational curve. Otherwise, it is the sum of two concurrent lines on X_r . We denote by \mathcal{K}_r the set of conic classes.

A *conic fibration* $X_r \rightarrow \mathbf{P}^1$ is given by the complete linear system of a conic on X_r . Hence, \mathcal{K}_r corresponds to the set of conic fibrations up to projective equivalence.

- (5). The orthogonal complement $K^\perp = \{ \alpha \in \text{Pic}(X_r) \mid \alpha \cdot K = 0 \}$ is free of rank r and spanned by the classes

$$\alpha_i = e_i - e_{i+1} \quad \text{for } i = 1, \dots, r-1 \quad \text{and} \quad \alpha_r = 3h - e_1 - e_2 - e_3.$$

Together with the positive definite symmetric form $-(\cdot, \cdot)|_{K^\perp}$ coming from the intersection pairing, the α_i 's define a root system of type E_r , with the convention that $E_4 = A_4$, $E_5 = D_5$, see the following figure:

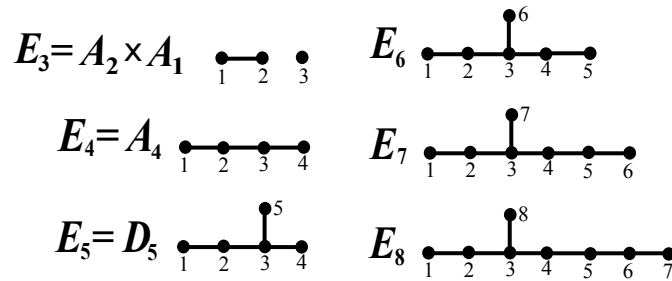


FIGURE 1. Dynkin diagrams E_r (with i standing for α_i for any $i = 1, \dots, r$)

- (6). For any $i = 1, \dots, r$, the map
- $$(7) \quad s_{\alpha_i} : \beta \mapsto \beta + (\beta \cdot \alpha_i) \alpha_i$$

is an involutive automorphism of $(\text{Pic}(X_r), (\cdot, \cdot))$ which lets K invariant. The restrictions of the s_{α_i} 's to $R_r = K^\perp \otimes_{\mathbf{Z}} \mathbf{R}$ are orthogonal reflections and they generate a Weyl group of type E_r , denoted by W_r . In particular, W_r is finite.

- (7). For simplicity, we set $s_i = s_{\alpha_i}$ for any i . When $i = 1, \dots, r-1$, the reflection s_i acts on $\text{Pic}(X_r)$ by interchanging e_i with e_{i+1} , leaving other exceptional classes e_k and h fixed. The reflection s_r acts as a Cremona transformation, i.e., one has $s_r(h) = 2h - e_1 - e_2 - e_3$ and $s_r(e_i) = h - e_j - e_k$ for $\{i, j, k\} = \{1, 2, 3\}$ and s_r leaves e_k fixed for $k = 4, \dots, r$.
- (8). For an element $w \in W_r$, we denote $(-1)^w = (-1)^{l(w)} \in \{\pm 1\}$ the *signature* of w . Here $l(w)$ stands for the length of w which by definition is the smallest non negative integer m such that one can write $w = s_{i_1} \cdots s_{i_m}$ for some i_1, \dots, i_m in $\{1, \dots, r\}$. The map $W_r \rightarrow \{\pm 1\}$, $w \mapsto (-1)^w$ is a group morphism, called the *signature*. The associated *signature representation* is the unique non trivial representation of W_r of dimension 1.
- (9). Any line ℓ (respectively, any conic class c) on X_r belongs to the W_r -orbit of the exceptional divisor e_1 (respectively, $h - e_1$). This follows from Noether's inequality (e.g., see [Do2, p. 288]). Equivalently: W_r acts transitively on the set \mathcal{L}_r of lines (respectively, on the set \mathcal{K}_r of conic classes).
- (10). Any conic fibration $\phi_c : X_r \rightarrow \mathbf{P}^1$ corresponding to a conic class c has exactly $r-1$ reducible fibers, each a union of two lines intersecting at a point. In particular, each conic class is of the form $c = \ell + \ell'$, with ℓ, ℓ' lines such that $\ell \cdot \ell' = 1$. We will often write $\ell + \ell'$ to indicate the reducible conic $\ell \cup \ell'$.
- (11). For $r > 3$, the stabilizer W_{e_r} of $e_r \in \mathcal{L}_r$ is generated by the reflections s_i 's for i ranging from 1 to r and distinct from $r-1$. It follows that W_{e_r} is isomorphic to the Weyl group associated to the Dynkin diagram E'_{r-1} obtained by removing the $(r-1)$ -th node as well as the edge adjacent to it from E_r , that is $W_{e_r} \simeq W(E_{r-1})$. In particular, for $r > 3$ we have $l_r = |\mathcal{L}_r| = |W_r|/|W_{r-1}|$. For $r = 3$, one has $W_{e_3} = \langle s_1 \rangle \simeq \{\pm 1\}$ and $l_3 = |\mathcal{L}_3| = 6$.
- (12). The stabilizer W_{c_1} of $c_1 = h - e_1 \in \mathcal{K}$ is generated by the reflections s_2, \dots, s_r . This subgroup of W_r is isomorphic to the Weyl group associated to the Dynkin diagram E''_{r-1} obtained by removing the first node as well as the edge adjacent to it from E_r , which hence is of type D_{r-1} .⁴ In particular, we have $\kappa_r = |\mathcal{K}_r| = |W_r|/|W(D_{r-1})| = |W_r|/(2^{r-2}(r-1)!)$.
- (13). For any mutually disjoint $r-2$ lines $\ell_1, \dots, \ell_{r-2}$, there exists an element w of the Weyl group W such that $w \cdot e_i = \ell_i$ for all $i = 1, \dots, r-2$ (cf. Corollary 26.8.(i) in [Ma]).

Some numerical invariants associated to the Weyl groups W_r and the sets of lines and conics \mathcal{L}_r and \mathcal{K}_r are gathered in the following table:

An important ingredient in our approach is that for each case $r \in \{3, \dots, 8\}$, there are explicit descriptions of both sets \mathcal{L}_r and \mathcal{K}_r (when those are seen as subsets of $\text{Pic}(X_r)$). We mention only the case when $r = 8$ (from which the other cases can be easily deduced) and refer to [Ma, §26] and [Do1] for details and proofs.

When viewed as elements of $\text{Pic}(X_r)$, any line or conic class is uniquely determined by the tuple of its integer coordinates $(d, m_1, \dots, m_r) \in \mathbf{Z}^{r+1}$ with respect to the basis $(h, -e_1, \dots, -e_r)$ of the Picard lattice. Let the *type* of a coordinate $(r+1)$ -tuple (d, m_1, \dots, m_r) by a symbol $(d; k_1^{n_1}, \dots, k_s^{n_s})$ for some integers $k_t \neq 0$ and $n_t > 0$ for $t = 1, \dots, s \leq r$, with the defining property that among the non zero m_1, \dots, m_r , exactly n_t are equal to k_t , this for all t ranging from 1 to s (for example, the type of $(6, 2, 2, 2, 3, 2, 2, 2, 0) \in \mathbf{Z}^9$ is $(6; 3, 2^6)$, etc).

⁴Here we use the convention that $D_2 = A_1 \times A_1$ and $D_3 = A_3$.

r	3	4	5	6	7	8
E_r	$A_2 \times A_1$	A_4	D_5	E_6	E_7	E_8
$W_r = W(E_r)$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	\mathfrak{S}_5	$(\mathbf{Z}/2\mathbf{Z})^4 \rtimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\omega_r = W_r $	12	5!	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$l_r = \mathcal{L}_r $	6	10	16	27	56	240
$\kappa_r = \mathcal{K}_r $	3	5	10	27	126	2160

TABLE 1.

In the table below, we list all the types of lines and conics classes on X_8 , and indicate the number of classes there are for each type (see [Ma, Prop. 26.1] and [Do1, §8.8]).

Lines on X_8		Conic classes on X_8	
Types	Number of such	Types	Number of such
(0; -1)	8	(1; 1)	8
(1; 1 ²)	28	(2; 1 ⁴)	70
(2; 1 ⁵)	56	(3; 2, 1 ⁵)	168
(3; 2, 1 ⁶)	56	(4; 2 ³ , 1 ⁴)	280
(4; 2 ³ , 1 ⁵)	56	(4; 3, 1 ⁷)	8
(5; 2 ⁶ , 1 ²)	28	(5; 2 ⁶ , 1)	56
(6; 3, 2 ⁷)	8	(5; 3, 2 ³ , 1 ⁴)	280
		(6; 3 ² , 2 ⁴ , 1 ²)	420
		(7; 3 ⁴ , 2 ³ , 1)	280
		(7; 4, 3, 2 ⁶)	56
		(8; 3 ⁷ , 1)	8
		(8; 4, 3 ⁴ , 2 ³)	280
		(9; 4 ² , 3 ⁵ , 2)	168
		(10; 4 ⁴ , 3 ⁴)	70
		(11; 4 ⁷ , 3)	8

TABLE 2. Types of lines and of conic classes and their numbers for the degree 1 del Pezzo surface X_8 .

3. The identity $\mathbf{H}\text{Log}^{r-2}$

In the whole section, we fix $3 \leq r \leq 8$. For most of the time, we will denote for simplicity

$$X = X_r, \quad K = K_{X_r}, \quad \mathcal{L} = \mathcal{L}_r, \quad \mathcal{K} = \mathcal{K}_r, \quad \text{etc.}$$

3.1. For each conic class $c \in \mathcal{K}$, we consider the corresponding conic fibration $\phi_c : X \rightarrow \mathbf{P}^1$ and we denote by $\Sigma_c \subset \mathbf{P}^1$ the set of $r - 1$ distinct points corresponding to the reducible fibers of ϕ_c . We may assume without loss of generality that $\Sigma_c = \{\sigma_c^1, \dots, \sigma_c^{r-1}\}$ (with $\sigma_c^{r-1} = \infty$), i.e., we are in the situation of a web of hypersurfaces (conics, in our case) as described in §2.1.5. We consider the same set-up and notations as in §2.1.5: we have

- $Y = X \setminus L$ with $L = L_r = \sum_{\ell \in \mathcal{L}} \ell \subset X$, and $\mathcal{H} = \mathbf{H}^0(X, \Omega_X^1(\text{Log } L))$;
- and for any conic class $c \in \mathcal{K}$, we set:
 - $\mathcal{H}_{\Sigma_c} = \mathbf{H}^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^1(\text{Log } \Sigma_c))$ and $\mathcal{H}_c = \phi_c^* \mathcal{H}_{\Sigma_c} \subset \mathcal{H}$;
 - $\eta'_c = \wedge_{i=1}^{r-2} \left(\frac{dz}{z - \sigma_c^i} \right) \in \wedge^{r-2} \mathcal{H}_{\Sigma_c} \subset (\mathcal{H}_{\Sigma_c})^{\otimes r-2}$; and
 - $\eta_c = \wedge_{i=1}^{r-2} \left(\frac{d\phi_c}{\phi_c - \sigma_c^i} \right) = \phi_c^* \eta'_c \in \wedge^{r-2} \mathcal{H}_c \subset (\mathcal{H}_c)^{\otimes r-2}$.

Each of the elements η_c generate the 1-dimensional \mathbf{C} -vector spaces $\wedge^{r-2} \mathcal{H}_c$ and is canonically defined up to sign. In what follows, we identify $\wedge^{r-2} \mathcal{H}_c$ with its image in $\wedge^{r-2} \mathcal{H}$.

Using the notations of §2.1.5 for any $y \in Y$ and as holomorphic germs at this point, one has

$$(8) \quad AI_c^{r-2}(\phi_c) = AI_{\Sigma_c, y}^{r-2}(\phi_c) = \Pi_y(\eta_c) \in \mathcal{O}_{Y, y}.$$

It then follows from Lemma 2.2, that Theorem 1.1 is equivalent to the following statement:

Theorem 3.1. 1. *Up to a global sign, there is a canonical choice of a tuple $(\tau_c)_{c \in \mathcal{K}}$ with $\tau_c = \pm \eta_c$ for each $c \in \mathcal{K}$ and such that the following equality holds true in $\wedge^{r-2} \mathcal{H}$:*

$$(9) \quad \sum_{c \in \mathcal{K}_r} \tau_c = 0$$

2. *Moreover, the identity (9) spans the space of linear relations between the τ_c 's, i.e., if $(c_c)_{c \in \mathcal{K}} \in \mathbf{C}^{\mathcal{K}}$ is such that $\sum_{c \in \mathcal{K}_r} c_c \tau_c = 0$ then all the c_c 's are equal.*

The rest of this section is devoted to proving this result.

3.2. The irreducible components of L being the lines $\ell \in \mathcal{L}$, one can define a Poincaré residue map $\text{Res}_L = \oplus_{\ell \in \mathcal{L}} \text{Res}_\ell : \Omega_X^1(\text{Log } L) \rightarrow \oplus_{\ell \in \mathcal{L}} \mathcal{O}_\ell$ wich makes the following sequence of sheaves exact:

$$0 \rightarrow \Omega_X^1 \mapsto \Omega_X^1(\log L) \mapsto \oplus_{\ell \in \mathcal{L}} \mathcal{O}_\ell \rightarrow 0.$$

As X is a rational variety, we have that $\mathbf{H}^0(X, \Omega_X^1) = 0$, hence the residue map induces an injective map of \mathbf{C} -linear vector spaces $\text{Res}_L : \mathcal{H} \hookrightarrow \mathbf{C}^{\mathcal{L}}$ and in turn an injective linear map $\wedge^{r-2} \mathcal{H} \hookrightarrow \wedge^{r-2} \mathbf{C}^{\mathcal{L}}$.

Given a conic fibration $\phi_c : X \rightarrow \mathbf{P}^1$ associated to a conic class $c \in \mathcal{K}$, we denote by C_c^1, \dots, C_c^{r-1} the reducible fibers of ϕ_c , with $C_c^i = \phi_c^{-1}(\sigma_c^i)$ for $i = 1, \dots, r - 1$ (with $\sigma_c^{r-1} = \infty$). Each conic C_c^i is a union of two lines $\ell_c^i, \tilde{\ell}_c^i$ intersecting in one point. It follows that the residues

$$\text{Res}_L(d\phi_c / (\phi_c - \sigma_c^i)) = C_c^i - C_c^{r-1} = (\ell_c^i + \tilde{\ell}_c^i) - (\ell_c^{r-1} + \tilde{\ell}_c^{r-1}) \in \mathbf{C}^{\mathcal{L}}$$

for $i = 1, \dots, r-2$, form a basis for the image of $\mathcal{H}_c \subset \mathcal{H}$ under the injective map $\mathcal{H} \hookrightarrow \mathbf{C}^{\mathcal{L}}$. Consequently, we get that the image of $\eta_c \in \wedge^{r-2} \mathcal{H}_c \subset \wedge^{r-2} \mathcal{H}$ under the injective map $\wedge^{r-2} \mathcal{H} \hookrightarrow \wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ is

$$(10) \quad (C_c^1 - C_c^{r-1}) \wedge \dots \wedge (C_c^{r-2} - C_c^{r-1}).$$

3.3. The Weyl group W acts on the set of lines \mathcal{L} and on the set of conic classes \mathcal{K} in a compatible way. In particular, for $c \in \mathcal{K}$ given, the action of any $w \in W$ sends the reducible fibers of ϕ_c to the reducible fibers of $\phi_{w \cdot c}$ in the following way:

$$w \cdot C_c^i = w \cdot \ell_c^i + w \cdot \tilde{\ell}_c^i.$$

On the other hand, the action of W on \mathcal{L} induces a canonical linear action of W on $\mathbf{C}^{\mathcal{L}}$, and therefore on $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$. The action of $w \in W$ on any wedge product $\wedge_{i=1}^{r-2} \ell_i$ with $(\ell_i)_{i=1}^{r-2} \in \mathcal{L}^{r-2}$ is given by

$$w \cdot (\ell_1 \wedge \dots \wedge \ell_{r-2}) = (w \cdot \ell_1) \wedge \dots \wedge (w \cdot \ell_{r-2}).$$

3.4. We now fix a base conic class $c_1 = h - e_1$ and label the reducible fibers of the associated conic fibration $\phi_{c_1} : X \rightarrow \mathbf{P}^1$ by

$$C^i = C_{c_1}^{i-1} = l_{1i} + e_i \quad \text{for } i = 2, \dots, r,$$

where l_{1i} stands for the class of the strict transform under the blow-up map β of the line in \mathbf{P}^2 through p_1 and p_i , i.e., $l_{1i} = h - e_1 - e_i$. As a generator of (the image in $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ of) $\wedge^{r-2} \mathcal{H}_{c_1}$, we choose and fix

$$\tau_{c_1} = (C^2 - C^r) \wedge (C^3 - C^r) \wedge \dots \wedge (C^{r-1} - C^r) \in \wedge^{r-2} \mathbf{C}^{\mathcal{L}}.$$

For $w \in W$ arbitrary, we have:

$$(11) \quad w \cdot \tau_{c_1} = (w \cdot C^2 - w \cdot C^r) \wedge (w \cdot C^3 - w \cdot C^r) \wedge \dots \wedge (w \cdot C^{r-1} - w \cdot C^r) \in \wedge^{r-2} \mathbf{C}^{\mathcal{L}}.$$

The stabilizer W_{c_1} of c_1 is a subgroup of W hence naturally acts on $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$. This action lets $\wedge^{r-2} \mathcal{H}_{c_1} \subset \wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ invariant hence $\wedge^{r-2} \mathcal{H}_{c_1}$ is naturally a W_{c_1} -representation (of dimension 1).

Lemma 3.2. 1. As a W_{c_1} -representation, $\wedge^{r-2} \mathcal{H}_{c_1}$ is isomorphic to the signature representation.

2. For $w \in W$ and $c \in \mathcal{K}$ such that $c = w \cdot c_1$, the element

$$\tau_c = (-1)^w (w \cdot \tau_{c_1})$$

is a well defined generator of $\wedge^{r-2} \mathcal{H}_c \subset \wedge^{r-2} \mathbf{C}^{\mathcal{L}}$.

With the notation of this lemma, since the $w \cdot C^i$'s for $i = 2, \dots, r-1$ are the non irreducible fibers of ϕ_c , one clearly has that τ_c coincides with η_c up to sign, hence, in particular, is a generator of $\wedge^{r-2} \mathcal{H}_c$. The interest of the second statement in this lemma is that it asserts that τ_c only depends on c and not on w (once τ_{c_1} has been fixed).

Proof. Proving 1. is elementary. Indeed, being a W_{c_1} -representation of dimension 1, there are only two possibilities for $\wedge^{r-2} \mathcal{H}_{c_1}$: either it is the trivial W_{c_1} -representation, or it is the signature representation. To prove that the second case does occur, it suffices to exhibit an element $w \in W_{c_1}$ such that $w \cdot \tau_{c_1} = -\tau_{c_1}$. Using (7) and (11), it is straightforward to check that any of the generators s_2, \dots, s_r of W_{c_1} (cf. §2.2.(12)) has this property.

The second part of the lemma follows easily from the first: for $w_1, w_2 \in W$ such that $w_1 \cdot c_1 = w_2 \cdot c_1 = c$, one has $w_2^{-1}w_1 \in W_{c_1}$, hence $w_2^{-1}w_1 \cdot \tau_{c_1} = (-1)^{w_2^{-1}w_1} \tau_{c_1}$ by 1. Thus $w_2 \cdot (w_2^{-1}w_1 \cdot \tau_{c_1}) = (-1)^{w_2^{-1}w_1} w_2 \cdot \tau_{c_1}$. The signature $w \mapsto (-1)^w$ being a group morphism, one has $(-1)^{w^{-1}} = (-1)^w$ for any w , therefore one obtains that $w_1 \cdot \tau_{c_1} = (-1)^{w_2} (-1)^{w_1} w_2 \cdot \tau_{c_1}$. This is equivalent to $(-1)^{w_1} w_1 \cdot \tau_{c_1} = (-1)^{w_2} w_2 \cdot \tau_{c_1}$, which is the relation ensuring that 2. holds true. \square

3.5. Proof of Theorem 3.1. We are going to prove that the following statements are satisfied:

1. The \mathcal{K} -tuple $(\tau_c)_{c \in \mathcal{K}}$ is a basis of $\bigoplus_{c \in \mathcal{K}} \wedge^{r-2} \mathcal{H}_c$ which is canonical, up to a global sign.
2. The sum $\sum_{c \in \mathcal{K}} \tau_c$ in $\wedge^{r-2} \mathcal{H}$ transforms as the signature under the action of W .
3. One has $\sum_{c \in \mathcal{K}} \tau_c = 0$ in $\wedge^{r-2} \mathcal{H}$.
4. Any scalar linear relation between the τ_c 's in $\wedge^{r-2} \mathcal{H}$ is a multiple of the one corresponding to the identity of 3.

The first assertion follows easily from the second part of Lemma 3.2 and from the fact that each τ_c necessarily coincides with η_c up to sign (details are left to the reader). In this subsection, we are going to establish first 2. (cf. Lemma 3.5) then 3. and 4. which will follow in the same time from Lemma 3.7.

For $k \geq 1$, following Manin (cf. [Ma, §26]), we call an *exceptional k -tuple* any k -tuple $(\ell_i)_{i=1}^k \in \mathcal{L}^k$ of non intersecting lines, i.e., such that $\ell_i \cdot \ell_j = 0$ for any i, j such that $1 \leq i < j \leq k$. Our approach to prove both 2. and 3. together is elementary and relies on the following

Lemma 3.3. *Let \mathcal{E} be an exceptional $(r-2)$ -tuple. There are exactly two conic classes such that each element of \mathcal{E} appears as a component of a reducible fiber of the associated conic fibration.*

Proof. Since W acts transitively on the set of exceptional $(r-2)$ -tuples (according to §2.2.(13)), one can assume that $\mathcal{E} = (e_3, \dots, e_r)$. Concretely, one wants to determine the conic classes $c \in \mathcal{K}$ such that $c - e_i \in \mathcal{L}$ for $i = 3, \dots, r$. Since the two sets \mathcal{L} and \mathcal{K} are finite and can be explicitly described (See Table 2 above for the case $r = 8$), the claim can be checked by a straightforward case by case verification. One finds that only the two conic classes $c_1 = h - e_1$ and $c_2 = h - e_2$ satisfy the above conditions. \square

We consider the following element of $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$:

$$\mathbf{hlog} = \mathbf{hlog}^{r-2} = \sum_{c \in \mathcal{K}} \tau_c.$$

We now prove that this element is equal to zero, by decomposing it in the canonical basis of $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ given by the wedge products $\ell_1 \wedge \ell_2 \wedge \dots \wedge \ell_{r-2}$ of $r-2$ pairwise distinct lines $\ell_1, \dots, \ell_{r-2} \in \mathcal{L}$.

Let $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$ be the proper subspace of $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ spanned by the wedge products $\ell_1 \wedge \ell_2 \wedge \dots \wedge \ell_{r-2}$ for all exceptional $(r-2)$ -tuple of lines $(\ell_i)_{i=1}^{r-2}$. We call such wedge products *exceptional*. We fix once for all a basis of $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$ formed by exceptional wedge products and call it the *exceptional basis* of $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$. Note that this basis is unique but only up to changing the signs of its elements. However, the arguments below are not essentially affected by this ambiguity.

We may assume that $\wedge_{i=3}^r e_i$ is an element of the exceptional basis and we denote by

$$(12) \quad \Lambda : \wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}} \longrightarrow \mathbf{C}$$

the associated linear coordinate form with respect to the exceptional basis we are working with, i.e., Λ is the linear form on $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$ characterized by the relations $\Lambda(\wedge_{i=3}^r e_i) = 1$ and $\Lambda(\wedge_{i=3}^r \ell_i) = 0$ for any exceptional wedge product $\wedge_{i=3}^r \ell_i$ such that $\{\ell_i\}_{i=3}^r \neq \{e_i\}_{i=3}^r$.

We first prove the

Lemma 3.4. *For any $c \in \mathcal{K}$, τ_c belongs to $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$, therefore $\mathbf{hlog} = \sum_{c \in \mathcal{K}} \tau_c$ as well.*

Proof. For $c \in \mathcal{K}$, τ_c is equal to (10) up to sign. The lemma follows easily by noticing that the conics C_c^1, \dots, C_c^{r-1} are pairwise disjoint and because each of them is the sum of two lines. \square

We also need the following

Lemma 3.5. *For any $w \in W$, one has $w \cdot \mathbf{hlog} = (-1)^w \mathbf{hlog}$.*

Proof. For a conic class $c \in \mathcal{K}$, let $w_c \in W$ be such that $c = w_c \cdot c_1$. By the definition of τ_c , one has

$$\tau_c = (-1)^{w_c} w_c \cdot \tau_{c_1} = (-1)^{w_c} \wedge_{i=2}^{r-1} (w_c \cdot C^i - w_c \cdot C^r).$$

Let w be an arbitrary element of W . From (11) and because $ww_c \cdot c_1 = w \cdot c$, it comes that

$$w \cdot \tau_c = (-1)^{w_c} \wedge_{i=2}^{r-1} (ww_c \cdot C^i - ww_c \cdot C^r) = (-1)^w \tau_{w \cdot c}.$$

Summing up on the conic classes and because $c \mapsto w \cdot c$ is a bijection of \mathcal{K} , one gets

$$w \cdot \mathbf{hlog} = \sum_{c \in \mathcal{K}} w \cdot \tau_c = (-1)^w \sum_{c \in \mathcal{K}} \tau_{w \cdot c} = (-1)^w \mathbf{hlog}. \quad \square$$

Since the W -action on $\text{Pic}(X)$ preserves the intersection product, $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$ is a proper W -submodule of $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$. Furthermore, W acts on $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$ by permuting the elements of the exceptional basis. Moreover, by §2.2.(13) this action on the exceptional wedge products is transitive (up to sign). To check that $\mathbf{hlog} = 0$ in $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$, it suffices to check that any element $\wedge_{i=1}^{r-2} \ell_i$ in the exceptional basis appears in \mathbf{hlog} with coefficient zero. For such $\wedge_{i=1}^{r-2} \ell_i$, let $w \in W$ be such that $w \cdot (\wedge_{i=3}^r e_i) = \wedge_{i=1}^{r-2} \ell_i$. It suffices to check that $\Lambda(w^{-1} \cdot \mathbf{hlog}) = 0$ (see notation (12)). By Lemma 3.5, this would follow from verifying that $\Lambda(\mathbf{hlog}) = 0$.

From Lemma (3.3), the only conic classes $c \in \mathcal{K}$ for which $\wedge_{i=3}^r e_i$ appears with non-zero coefficient in the decomposition of τ_c in the exceptional basis are $c_1 = h - e_1$ and $c_2 = h - e_2$, i.e.,

$$(13) \quad \{c \in \mathcal{K} \mid \Lambda(c) \neq 0\} = \{c_1, c_2\}.$$

Consequently, one has

$$(14) \quad \Lambda(\mathbf{hlog}) = \sum_{c \in \mathcal{K}} \Lambda(\tau_c) = \Lambda(\tau_{c_1}) + \Lambda(\tau_{c_2}).$$

On the other hand, considering our initial choice for τ_{c_1} , we have (see §3.2 above)

$$\tau_{c_1} = (\ell_{12} + e_2 - \ell_{1r} - e_r) \wedge (\ell_{13} + e_3 - \ell_{1r} - e_r) \wedge \dots \wedge (\ell_{1r-1} + e_{r-1} - \ell_{1r} - e_r)$$

from which we have immediately that

$$(15) \quad \Lambda(\tau_{c_1}) = \Lambda(-e_r \wedge e_3 \wedge e_4 \wedge \dots \wedge e_{r-1}) = (-1)^r.$$

From §2.2.(7), we know that the first fundamental reflection s_1 acts as the transposition exchanging e_1 and e_2 on the set $\{h, e_1, \dots, e_r\}$. Therefore, one has $s_1 \cdot c_1 = c_2$ and $s_1 \cdot (e_3 \wedge \dots \wedge e_r) = (s_1 \cdot e_3) \wedge \dots \wedge (s_1 \cdot e_r) = e_3 \wedge \dots \wedge e_r$ which implies that $\Lambda(s_1 \cdot \tau_{c_1}) = \Lambda(\tau_{c_1})$. Since $\tau_{c_2} = (-1)^{s_1} s_1 \cdot \tau_{c_1}$ by definition and because $(-1)^{s_1} = -1$, one obtains that

$$(16) \quad \Lambda(\tau_{c_2}) = (-1)^{r-1} = -\Lambda(\tau_{c_1}).$$

Substituting (15) and (16) in (14) gives $\Lambda(\mathbf{hlog}) = 0$ which, as explained above, implies the

Lemma 3.6. *One has $\mathbf{hlog} = 0$.*

The assertion 3. at the beginning of §3.5 is proved. We now prove assertion 4. Consider the graph with vertices in \mathcal{K} , with $c, c' \in \mathcal{K}$ joined by an edge if there exists an element $\wedge_{i=1}^{r-2} \ell_i$ of the exceptional basis that appears with non-zero coefficient in the decomposition of both τ_c and $\tau_{c'}$ in the exceptional basis. We have

Lemma 3.7. *For any element $\ell_1 \wedge \dots \wedge \ell_{r-2}$ of the exceptional basis, there are exactly two conic classes $c, c' \in \mathcal{K}$ such that $\ell_1 \wedge \dots \wedge \ell_{r-2}$ appears with non-zero coefficient in the decomposition of τ_c and $\tau_{c'}$ in the exceptional basis. Moreover, these two coefficients are opposite.*

Proof. For $e_3 \wedge \dots \wedge e_r$, this has been proved above (cf. (13), (15) and (16)). The general case follows by considering the action of W . \square

Assume now that there exists a relation $\sum_{c \in \mathcal{K}} c_c \tau_c = 0$ for some $c_c \in \mathbf{C}$. By Lemma 3.7, for two conic classes c and c' connected by an edge, we have $c_c + c_{c'} = 0$. Hence it suffices to prove that our graph is connected. As the action of W on \mathcal{K} is transitive, it suffices to check that for all $w \in W$ the classes $c_1 = h - e_1$ and $w \cdot c_1$ are connected by a sequence of edges. Furthermore, using again the action of W , it suffices to check this for $w = s_i$ ($i = 1, \dots, r$). The reflections s_2, \dots, s_r belong to the stabilizer of c_1 , so there is nothing to prove. If $w = s_1$, then $c_2 = w \cdot c_1 = h - e_2$. As proved above, c_1 and c_2 are connected by an edge. The proof of Theorem 3.1 is now complete.

3.6. A representation-theoretic interpretation. The \mathcal{K} -tuple $\tau_{\mathcal{K}} = (\tau_c)_{c \in \mathcal{K}}$ is an algebraic avatar of the κ_r -tuple of hyperlogarithms $(\epsilon_i AI_i^{r-2}(U_i))_{i=1}^{\kappa_r}$ involved in the statement of Theorem 1.1. It turns out that $\tau_{\mathcal{K}}$ as well as the fact that the identity

$$(17) \quad \mathbf{hlog}^{r-2} = \sum_{c \in \mathcal{K}} \tau_c = 0$$

is satisfied in $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ can be interpreted within the representation theory of the Weyl group W . This subsection is devoted to an exposition of this. For details, we refer to [Pi] where the second author used this approach to give a representation theoretic proof of Theorem 3.1 for del Pezzo surfaces of degree $d \in \{2, \dots, 6\}$.

The key points from this perspective are the following (cf. [Pi, §3.2] for details):

1. one can define a natural action of W on the direct sum $\bigoplus_{c \in \mathcal{K}} \wedge^{r-2} \mathcal{H}_c$ such that the map

$$\iota_{\mathcal{K}} : \bigoplus_{c \in \mathcal{K}} \wedge^{r-2} \mathcal{H}_c \longrightarrow \wedge^{r-2} \mathbf{C}^{\mathcal{L}}$$

induced by the natural inclusion $\wedge^{r-2} \mathcal{H}_c \hookrightarrow \wedge^{r-2} \mathcal{H} \xrightarrow{\text{Res}_L} \wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ becomes a morphism of W -representations;

2. as a W -representation, $\bigoplus_{c \in \mathcal{K}} \wedge^{r-2} \mathcal{H}_c$ is isomorphic to $\mathbf{sign} \otimes \mathbf{C}^{\mathcal{K}}$ where \mathbf{sign} stands for the signature W -representation and where the W -module structure on $\mathbf{C}^{\mathcal{K}}$ is the one induced by the action of W on \mathcal{K} by permutations;
3. the span of $\tau_{\mathcal{K}} = (\tau_c)_{c \in \mathcal{K}}$ is W -invariant and is the unique 1-dimensional irreducible component of $\bigoplus_{c \in \mathcal{K}} \wedge^{r-2} \mathcal{H}_c$ which is isomorphic to the signature representation \mathbf{sign} ;
4. from 1. and 3. it follows that $\iota_{\mathcal{K}}(\tau_{\mathcal{K}}) = \sum_c \tau_c = \mathbf{hlog}^{r-2}$ spans a W -subrepresentation of $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ which either is zero or is isomorphic to \mathbf{sign} ;
5. the decomposition of $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ in W -irreducibles can be determined explicitly (by means of computations with GAP). In particular, \mathbf{sign} appears with positive multiplicity in this decomposition if and only if $r = 8$.

From the points 4. and 5. above, one obtains an alternative, conceptual proof of the identity (9) of Theorem 3.1. This proof relies on the decompositions of $\wedge^{r-2} \mathbf{C}^{\mathcal{L}}$ in irreducible W -modules which are interesting on their own and appear to be new for $r > 4$ (see [Pi, Proposition 3.2]). Note, for $r = 8$ one would need to adapt this approach in order to prove our main result, possibly by considering the subrepresentation given by $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$ (Lemma 3.4). More generally, it would be interesting to determine the decomposition into irreducibles of $\wedge_{exc}^{r-2} \mathbf{C}^{\mathcal{L}}$ for any r and to verify whether it admits the signature as one of its irreducible components or not (we know that it is not the case for $r \leq 7$).

3.7. The identity \mathbf{HLog}^{r-2} is defined over \mathbf{Z} . By requiring that all the residues considered are integers, one defines canonical \mathbf{Z} -structures $\mathcal{H}^{\mathbf{Z}}, \mathcal{H}_{\Sigma_c}^{\mathbf{Z}}, \mathcal{H}_c^{\mathbf{Z}}$ on the spaces $\mathcal{H}, \mathcal{H}_{\Sigma_c}, \mathcal{H}_c$ respectively, which are compatible with respect to pull-backs and inclusions, i.e., one has $\phi_c^* \mathcal{H}_{\Sigma_c}^{\mathbf{Z}} = \mathcal{H}_c^{\mathbf{Z}}$ and $\mathcal{H}_c \subset \mathcal{H}$ induces an inclusion $\mathcal{H}_c^{\mathbf{Z}} \subset \mathcal{H}^{\mathbf{Z}}$ for any $c \in \mathcal{K}$. Moreover, the residue map Res_L of §3.2 admits a canonical lift $\mathcal{H}^{\mathbf{Z}} \rightarrow \mathbf{Z}^{\mathcal{L}}$ over \mathbf{Z} . We leave it to the reader to verify that all the statements in §3.3–§3.6 hold over \mathbf{Z} . In conclusion, \mathbf{HLog}^{r-2} is defined over \mathbf{Z} , a fact which may be interesting from an arithmetic perspective.

3.8. For any $n \geq 2$, it is tempting to consider more generally blow-ups $Y_r = \text{Bl}_r(\mathbf{P}^n)$ at $r \geq n + 2$ general points and attempt to generalize Theorem 3.1 by following the exact same approach as in this section. One defines a symmetric bilinear form (\cdot, \cdot) on $K_{Y_r}^{\perp} \subset \text{Pic}(Y_r) = \mathbf{Z}\{H, E_1, \dots, E_r\}$ (where H is the hyperplane class and the E_i 's are the classes of the exceptional divisors) by setting

$$(H, H) = n - 1, \quad (H, E_i) = 0 \quad \text{and} \quad (E_i, E_j) = -\delta_{ij} \quad \text{for } i, j = 1, \dots, r.$$

One can define a Coxeter group W associated to a T-shaped Dynkin-type diagram $T_{2, n+1, r-n-1}$ such that W acts on $\text{Pic}(Y_r)$ in a geometric fashion, in particular preserving the bilinear form (\cdot, \cdot) . It is known that W is finite if and only if

$$(18) \quad \frac{1}{n+1} + \frac{1}{r-n-1} > \frac{1}{2}.$$

This condition is equivalent to Y_r being a *Mori dream space* [Mu, CT] and translates to $r \leq n + 3$ if $n \geq 5$, $r \leq 8$ if $n = 4, 2$, and $r \leq 7$ if $n = 3$. Such blow-ups are natural generalizations of del Pezzo surfaces. We refer to [Mu, CT] for more details.

Assume that (18) is satisfied. One can replace the set of lines \mathcal{L} by the set of *Weyl divisors*, i.e., divisors in the finite orbit $W \cdot E_1$, and the set \mathcal{K} of conic classes with the set of *Weyl pencils*, which we define as one-dimensional linear systems $|E + F|$, for E, F Weyl divisors such that $(E, F) = 1$. Such a linear system \mathfrak{c} induces a rational map $\phi_{\mathfrak{c}} : Y_r \dashrightarrow \mathbf{P}^1$ which has $r - n + 1$ reducible fibers, with components E', F' Weyl divisors such that $(E', F') = 1$ and $E' + F' = E + F$ in $\text{Pic}(Y_r)$. It is straightforward to check that the group W acts transitively on the sets \mathcal{L} and \mathcal{K} . All the constructions leading up to Lemma 3.2 hold in this more general context, i.e., for each Weyl pencil $\mathfrak{c} \in \mathcal{K}$ we can construct canonical elements $\tau_{\mathfrak{c}} \in \wedge^{r-n} \mathbf{C}^{\mathcal{L}}$ (well-defined up to a global sign) such that Lemma 3.2 and the statements 1. and 2. at the beginning of §3.5 hold. However, if $n \geq 3$, the identity $\sum_{\mathfrak{c}} \tau_{\mathfrak{c}} = 0$ never holds. One can follow the same approach as in this section to prove an analogue of Lemma 3.3: for any *exceptional* $(r - n)$ -tuple \mathcal{E} there are exactly n Weyl pencils such that each element of \mathcal{E} appears as a component of a reducible fiber of the associated fibration. The analogue of Lemma 3.7 is that the coefficient with which an element $E_1 \wedge \dots \wedge E_{r-n}$ in the exceptional basis appears in $\sum_{\mathfrak{c} \in \mathcal{K}} \tau_{\mathfrak{c}}$ is (up to a sign) $(n - 2)$, hence, never zero if $n \geq 3$. In analytic terms, this translates as the fact that, if $AI_{\mathfrak{c}}^{r-n}$ stands for the hyperlogarithm on \mathbf{P}^1 such that $AI_{\mathfrak{c}}^{r-n}(\phi_{\mathfrak{c}}) = \Pi_{Y_r}^y(\tau_{\mathfrak{c}})$ for any $\mathfrak{c} \in \mathcal{K}$ (for a previously chosen base point y general in Y_r), then the functional identity $\sum_{\mathfrak{c} \in \mathcal{K}} AI_{\mathfrak{c}}^{r-n}(\phi_{\mathfrak{c}}) = 0$ is not satisfied in the vicinity of y on Y_r .

However, the space $\mathbf{HLog}_{Y_r}^{r-n}$ of tuples $(\alpha_{\mathfrak{c}})_{\mathfrak{c} \in \mathcal{K}} \in \mathbf{C}^{\mathcal{K}}$ such that $\sum_{\mathfrak{c} \in \mathcal{K}} \alpha_{\mathfrak{c}} AI_{\mathfrak{c}}^{r-n}(\phi_{\mathfrak{c}}) = 0$ is not trivial. Similar to the case of del Pezzo surfaces, for every exceptional r -tuple \mathcal{E} there exists a small modification $F_{\mathcal{E}} : Y_r \dashrightarrow \text{Bl}_{q_1, \dots, q_r}(\mathbf{P}^n)$, where $\beta : \text{Bl}_{q_1, \dots, q_r}(\mathbf{P}^n) \rightarrow \mathbf{P}^n$ is a blow-up of a (possibly distinct) configuration of points q_1, \dots, q_r in general position. Let $J \subset \{1, \dots, r\}$ be of cardinal $n - 2$ and let $\pi_J : \mathbf{P}^n \dashrightarrow \mathbf{P}^2$ be the linear projection from the $(n - 3)$ -plane in \mathbf{P}^n spanned by the q_j 's for $j \in J$. Setting $q'_k = \pi_J(q_k)$ for $k \notin J$, one has $r - n + 2$ points in general position in \mathbf{P}^2 . The total space of the blow-up $\beta_J : \text{Bl}_{\{q'_k\}_{k \notin J}}(\mathbf{P}^2) \rightarrow \mathbf{P}^2$ is a del Pezzo surface of degree $d = 7 - r + n$ which we will denote by $dP_{d,J}$. We let $\Pi_{\mathcal{E},J} : Y_r \dashrightarrow dP_{d,J}$ be the induced rational maps (so that one has $\beta_J \circ \Pi_{\mathcal{E},J} = \pi_J \circ \beta \circ F_{\mathcal{E}}$ as rational maps). If \mathcal{K}_J denotes the set of conic classes on $dP_{d,J}$ and $\psi_{\kappa} : dP_{d,J} \rightarrow \mathbf{P}^1$ denotes the associated conic fibration for $\kappa \in \mathcal{K}_J$, then the compositions $\phi_{\kappa} = \psi_{\kappa} \circ \Pi_{\mathcal{E},J} : Y_r \dashrightarrow \mathbf{P}^1$ are Weyl pencils (and all Weyl pencils are of this form). Consequently one obtains an injection $\mathcal{K}_J \subset \mathcal{K}$. If $\sum_{\kappa \in \mathcal{K}_J} \epsilon_{\kappa} AI_{\kappa}^{r-n}(\psi_{\kappa}) = 0$ stands for the identity on $dP_{d,J}$ given by Theorem 1.1 (with $\epsilon_{\kappa} \in \{\pm 1\}$ for any $\kappa \in \mathcal{K}_J$), one gets that the hyperlogarithmic identity $\sum_{\kappa \in \mathcal{K}_J} \epsilon_{\kappa} AI_{\kappa}^{r-n}(\phi_{\kappa}) = 0$ holds true locally at y on Y_r . This identity corresponds to a non-zero element of $\mathbf{HLog}_{Y_r}^{r-n}$, which we denote by $\mathbf{HLog}_{\mathcal{E},J}^{r-n}$. It follows that $\mathbf{HLog}_{Y_r}^{r-n}$ is not trivial.

It is natural to ask whether the span of the set of $\mathbf{HLog}_{\mathcal{E},J}^{r-n}$'s for all pairs (\mathcal{E}, J) as above coincides with the whole space $\mathbf{HLog}_{Y_r}^{r-n}$ or not. For the case when $r = n + 2$ with $n \geq 2$ arbitrary, this follows from computations in [Pe]. By direct computations, we have verified that it is the case as well for $(n, r) = (3, 6)$ and $(n, r) = (4, 7)$. We conjecture that this happens in all cases. If true, this would say that regarding functional identities satisfied by the complete antisymmetric hyperlogarithms $AI_{\mathfrak{c}}^{r-n}$ on Y_r , there is nothing new since everything come from the 2-dimensional del Pezzo hyperlogarithmic identity \mathbf{HLog}^{r-n} (up to pull-backs under the maps $\Pi_{\mathcal{E},J} : Y_r \dashrightarrow dP_{d,J}$).

4. The identity \mathbf{HLog}^3 in explicit form

The identity \mathbf{HLog}^2 is equivalent to Abel's relation ($\mathcal{A}b$) which is written in explicit form. One can make the other hyperlogarithmic identities \mathbf{HLog}^{r-2} explicit as well. We illustrate this with the case when $r = 5$.

Let X_5 stand for the blow-up of \mathbf{P}^2 at the following five points: $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$, $p_4 = [1 : 1 : 1]$ and $p_5 = [a : b : 1]$, for some parameters $a, b \in \mathbf{C}$ such that

$$ab(a-1)(b-1)(a-b) \neq 0,$$

a condition that we assume to be satisfied in what follows.

We make explicit the weight 3 hyperlogarithmic identity $\mathbf{HLog}(X_5)$ when expressed in the affine coordinates x, y corresponding to the affine embedding $\mathbf{C}^2 \hookrightarrow \mathbf{P}^2$, $(x, y) \mapsto [x : y : 1]$.

Relatively to the coordinates x, y , the conic fibrations on X_5 correspond on \mathbf{P}^2 to the following rational functions U_i (where P stands for the affine polynomial $P = (1-b)x - (1-a)y - (a-b)$):

$$\begin{aligned} U_1 &= x & U_2 &= \frac{1}{y} & U_3 &= \frac{y}{x} & U_4 &= \frac{x-y}{x-1} & U_5 &= \frac{b(a-x)}{ay-bx} \\ U_6 &= \frac{P}{(x-1)(y-b)} & U_7 &= \frac{(x-y)(y-b)}{yP} & U_8 &= \frac{xP}{(x-y)(x-a)} & U_9 &= \frac{y(x-a)}{x(y-b)} & U_{10} &= \frac{x(y-1)}{y(x-1)}. \end{aligned}$$

For any $i = 1, \dots, 10$, the set of $\lambda \in \mathbf{P}^1$ for which $U_i^{-1}(\lambda)$ is reducible has the form $\{0, 1, r_i, \infty\}$ where $r_i \in \mathbf{P}^1 \setminus \{0, 1, \infty\}$ is given by

$$\begin{aligned} r_1 &= a & r_2 &= \frac{1}{b} & r_3 &= \frac{b}{a} & r_4 &= \frac{a-b}{a-1} & r_5 &= \frac{b(a-1)}{a-b} \\ r_6 &= \frac{b-a}{b} & r_7 &= \frac{1}{1-a} & r_8 &= 1-b & r_9 &= \frac{1-a}{1-b} & r_{10} &= \frac{a(b-1)}{b(a-1)}. \end{aligned}$$

For a triple (a, b, c) of pairwise distinct points on \mathbf{C} and a given base point $\xi \in \mathbf{C} \setminus \{a, b, c\}$, we consider the weight 3 hyperlogarithm $L_{a,b,c}^\xi$ defined by

$$L_{a,b,c}^\xi(z) = \int_\xi^z \left(\int_\xi^{u_3} \left(\int_\xi^{u_2} \frac{du_1}{u_1-c} \right) \frac{du_2}{u_2-b} \right) \frac{du_3}{u_3-a}$$

for any z sufficiently close to ξ , and we denote by $AI_{a,b,c}^\xi$ its antisymmetrization:

$$(19) \quad AI_{a,b,c}^\xi = \frac{1}{6} \left(L_{a,b,c}^\xi - L_{a,c,b}^\xi - L_{b,a,c}^\xi + L_{b,c,a}^\xi + L_{c,a,b}^\xi - L_{c,b,a}^\xi \right).$$

We now fix a base point $\zeta \in \mathbf{C}^2$ image of a point in X_5 which does not belong to any line. For $i = 1, \dots, 10$, we set $\zeta_i = U_i(\zeta) \in \mathbf{C} \setminus \{0, 1, r_i\}$ and

$$AI_i^3 = AI_{0,1,r_i}^{\zeta_i}.$$

Then one can verify that $\mathbf{HLog}(X_5)$ has the following explicit form

$$(20) \quad \sum_{i=1}^{10} AI_i^3(U_i) = 0,$$

a functional identity which is satisfied on any sufficiently small neighborhood of ζ .

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