



HAL
open science

Expectile and M-quantile regression for panel data

Ian Meneghel Danilevycz, Valdério Anselmo Reisen, Pascal Bondon

► **To cite this version:**

Ian Meneghel Danilevycz, Valdério Anselmo Reisen, Pascal Bondon. Expectile and M-quantile regression for panel data. *Statistics and Computing*, 2024, 34 (3), pp.97. 10.1007/s11222-024-10396-7. hal-04508666

HAL Id: hal-04508666

<https://universite-paris-saclay.hal.science/hal-04508666>

Submitted on 22 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Expectile and M-quantile regression for panel data

Ian Meneghel Danilevicz^{*1, 2}, Valdério Anselmo Reisen^{1, 2, 3, 4} and Pascal Bondon²

¹Universidade Federal de Minas Gerais, Department of Statistics, Brazil.

²Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France.

³Universidade Federal do Espírito Santo, Graduate Program in Environmental Engineer, Graduate Program in Economics, Brazil.

⁴Universidade Federal da Bahia, Institute of Mathematics and Statistics, Brazil.

Abstract

Linear fixed effect models are a general way to fit panel or longitudinal data with a distinct intercept for each unit. Based on expectile and M-quantile approaches, we propose alternative regression estimation methods to estimate the parameters of linear fixed effect models. The estimation functions are penalized by the least absolute shrinkage and selection operator (LASSO) to reduce the dimensionality of the data. Some asymptotic properties of the estimators are established, and finite sample size investigations are conducted to verify the empirical performances of the estimation methods. The computational implementations of the procedures are discussed, and real economic panel data from the Organisation for Economic Cooperation and Development (OECD) are analyzed to show the usefulness of the methods in a practical problem.

Keywords: Quantile regression, Expectile, M-estimation, repeated measures, LASSO.

1 Introduction

Panel data, longitudinal data, and repeated measures are widespread in applied sciences such as econometrics, medicine, and engineering. They need be analyzed with models considering the dual source of variability between and within observational units. There are two main approaches to handling this type of data; in the random effects viewpoint, explanatory variables and individual effects are independent, and in the fixed effects standpoint, this assumption is relaxed. The linear mixed model (LMM) might combine these two approaches, but the explanatory variables are usually treated as fixed effects, and individual intercepts are assumed as random effects (Laird and Ware, 1982). Random coefficients and fixed intercepts are allowed, but they are exceptions in applied research. The fixed effects specification is more robust than the random effects model. However, analyzing fixed effects panel data is more difficult due to the increasing number of parameters with the sample size (Horowitz and Lee, 2004; You and Zhou, 2014).

The least absolute deviation regression (LADR) estimates the median effect of regressors on the dependent variable rather than the mean effect as in the least square regression. The quantile regression (QR), which generalizes LADR to any percentile of interest, is a favorable alternative

*Corresponding author, email: ian.meneghel-danilevicz @ inserm.fr

for at least three reasons. This method returns a general relationship between predictors and the response variable (it is not restricted to median and mean effects), it does not require Gaussian errors, and it is less sensitive to heteroscedastic data (Koenker and Bassett Jr., 1978) and (Koenker, 2005, page 41).

The combination of QR and LMM seems to be a natural and necessary strategy since many longitudinal data are heteroscedastic and non-Gaussian. Therefore, quantile regression with fixed effects (QRFE) and penalized quantile regression with fixed effects (PQRFE) are appropriate methods for panel data (Koenker, 2004). A detailed overview is available in Marino and Farcomeni (2015), but this field has made expressive progress in recent years (Galarza et al., 2020; Li et al., 2020; Galvão et al., 2020; Tian et al., 2021). It is also possible to combine QR with random effects, but this is outside this paper’s scope.

The estimation of quantiles involves the check function, which is synthetic, robust, and easy to interpret (Koenker and Bassett Jr., 1978). However, the check function shows numerical instability close to zero where it is non-derivable (Alfò et al., 2021), producing a non-monotonic behavior named “the quantile crossing problem”, which violates the basic definition of percentiles (Koenker, 2005, page 55). The quantile crossing might be worse when there is a curse of dimensionality, which arises if the number of covariates is very large (Breckling et al., 2001). Finally, extreme quantile estimates display a progressive bias (Waltrup et al., 2015). There are some methods to correct these problems, as expectile regression (ER) which is a weighted least squared (Newey and Powell, 1987), and M-quantile regression (MQR) which generalizes the M-estimation proposed by Huber (1964) to quantile context (Breckling and Chambers, 1988). These two alternative approaches produce quantiles of a distribution related to the true distribution, as shown by Jones (1994). ER used to be more efficient and more stable (the regression curves hardly cross each other), but is more biased than QR (Schnabel and Eilers, 2009). MQR balances the trade-off between QR robustness and ER efficiency, see Kim and Oh (2020). This balance is tuned by a constant c as in the classical M-estimation method proposed by Huber (1964), and the choice of c in MQR is discussed by Kokic et al. (2002); Kim and Oh (2020). MQR is also more resilient to curse of dimensionality (Kokic et al., 2002).

To the best of our knowledge, the papers that combine expectile regression with random effects (ERRE) are Waltrup and Kauermann (2017), and Li et al. (2022). Tzavidis et al. (2016); Alfò et al. (2021) propose M-quantile regression with random effects (MQRRE). Geraci and Bottai (2007, 2014) introduce quantile regression with random effects (QRRE). However, these works assume that the random intercepts have no location contribution, are uncorrelated with the explanatory predictors, and follow a Gaussian distribution. The original proposition of QRFE is free of all these restrictions (Koenker, 2004). QRFE treats individual intercepts as fixed effects that may be different from zero. An expectile approach for fixed effects is proposed by Barry et al. (2023), i.e., expectile regression with fixed effects (ERFE). The negative counterpart of QRFE and ERFE is that the number of parameters to be estimated increases with the number of subjects. Hence, PQRFE uses LASSO to shrink the dimension of the parameter space (Koenker, 2004). The optimal value of LASSO’s tuning parameter for this model is discussed by Lamarche (2010); Lamarche and Parker (2021).

Based on the above literature review, we propose here three variants of QRFE by introducing alternative loss functions and penalizing the intercepts with Lasso. All proposed methods use either the expectile or the M-quantile as the loss function. The methods are penalized expectile regression with fixed effects (PERFE), M-quantile regression with fixed effects (MQRFE) and penalized M-quantile regression with fixed effects (PMQRFE). In Section 3, we show that the empirical performance depends on the loss function, which is expected. For example, ERFE and PERFE present, in general, better performance in the extreme quantiles than the standard ER, QR and MQR methods. About robustness, PMQRFE outperforms the other techniques and thus appears as an alternative approach to estimating LMM with fixed effects.

The remainder of this paper is as follows. In Section 2, the model and the estimation methods are introduced. In Section 3, the Monte Carlo simulation results are presented, and the empirical performance of each method is analyzed. Section 4 is committed to real data application of interest in the recent economics literature. Section 5 summarizes our conclusions.

2 Model and estimation methods

2.1 The model

In panel data analysis, mixed models may involve either random or fixed effects. In the first case, the intercepts are represented by zero-mean random variables and are assumed to be independent of the error term. Therefore, random effects models may omit the location effect of intercepts and only include them in the variance structure, which needs to be chosen appropriately (Diggle, 1988). Fixed effects models represent the individual effects by parameters to be estimated. However, the number of parameters increases with the number of units, and their estimation becomes inefficient (Horowitz and Lee, 2004; You and Zhou, 2014).

We denote by $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{im_i})^T$ the vector of measurements taken at times t_{i1}, \dots, t_{im_i} of the i th subject for $i = 1, \dots, n$. Following Laird and Ware (1982); Diggle (1988), the LMM assumes that each measurement Y_{ij} of \mathbf{Y}_i , $j = 1, \dots, m_i$, follows

$$Y_{ij} = \mathbf{X}_{ij}\boldsymbol{\beta} + \alpha_i + U_{ij}, \quad i = 1, \dots, n \quad j = 1, \dots, m_i, \quad (1)$$

where \mathbf{X}_{ij} is a known $(1 \times d)$ design vector, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d)^T$ is a vector of d unknown but fixed parameters to be estimated, α_i is the individual intercept, and U_{ij} is a random variable. Model (1) can be written in a vector form as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\alpha} + \mathbf{U},$$

where $\mathbf{Y} = (\mathbf{Y}_1^T, \dots, \mathbf{Y}_n^T)^T$ is a vector of length $N = \sum_{i=1}^n m_i$. $\mathbf{X}_i = [\mathbf{X}_{i1}^T, \dots, \mathbf{X}_{im_i}^T]^T$ is a $(m_i \times d)$ matrix and $\mathbf{X} = [\mathbf{X}_1^T, \dots, \mathbf{X}_n^T]^T$ is a $(N \times d)$ matrix. $\mathbf{Z} = [\mathbf{Z}_1, \dots, \mathbf{Z}_n]$ is a $(N \times n)$ incident matrix which k th element of column \mathbf{Z}_i equals one if $m_{i-1} + 1 \leq k \leq m_i$ and zero otherwise, with $m_0 = 0$. When $m_i = m$ for all i , $\mathbf{Z} = \mathbf{I}_n \otimes \mathbf{1}_m$, where \otimes denotes the Kronecker product, \mathbf{I}_n is the $(n \times n)$ identity matrix and $\mathbf{1}_m$ is the m -vector of ones. $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T$ is the n -vector of individual intercepts. $\mathbf{U} = (\mathbf{U}_1^T, \dots, \mathbf{U}_n^T)^T$ is the N -vector of errors, where $\mathbf{U}_i^T = (U_{i1}, \dots, U_{im_i})$ and the random vectors $\mathbf{U}_1, \dots, \mathbf{U}_n$ are independent.

In this paper, we use a QR model with fixed effects to quantify the influence of \mathbf{X}_{ij} on the response Y_{ij} . This model is written as

$$Q_{Y_{ij}}(\tau | \mathbf{X}_{ij}) = \mathbf{X}_{ij}\boldsymbol{\beta}(\tau) + \alpha_i(\tau), \quad (2)$$

where $Q_{Y_{ij}}(\tau | \mathbf{X}_{ij})$ is the conditional τ -quantile of Y_{ij} given \mathbf{X}_{ij} with $\tau \in (0, 1)$. In (2), $\boldsymbol{\beta}(\tau)$ denotes the vector of common parameters, and $\alpha_i(\tau)$ is a quantile-specific individual effect, see Koenker (2005, Chapter 8) and Galvão et al. (2020).

2.2 Estimation methods: quantile, expectile and M-quantile approaches

Let $\boldsymbol{\alpha}(\tau) = (\alpha_1(\tau), \dots, \alpha_n(\tau))^T$. In Model (2), the estimate $(\tilde{\boldsymbol{\alpha}}(\tau), \tilde{\boldsymbol{\beta}}(\tau))$ of $(\boldsymbol{\alpha}(\tau), \boldsymbol{\beta}(\tau))$ is a solution of

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^n, \boldsymbol{\beta} \in \mathbb{R}^d} \sum_{i=1}^n \sum_{j=1}^{m_i} \rho_\tau(Y_{ij} - \mathbf{X}_{ij}\boldsymbol{\beta} - \alpha_i), \quad (3)$$

where, for all $u \in \mathbb{R}$, $\rho_\tau(u) = \psi_\tau(u) g(u)$ with $\psi_\tau(u) = |\tau - \mathbf{1}_{u \leq 0}|$ and $g(\cdot)$ a convex loss function. Here, we discuss three possible choices for $g(\cdot)$.

- i) The most common choice is the absolute loss function $g_1(u) = |u|$; then $\rho_\tau(\cdot)$ is the check function.
- ii) An alternative for $g(\cdot)$ is the squared loss function $g_2(u) = u^2$; then $\rho_\tau(\cdot)$ is the asymmetric least squared function,
- iii) The Huber loss function $g_3(u) = (c|u| - \frac{1}{2}c^2)\mathbb{1}_{|u|>c} + \frac{1}{2}u^2\mathbb{1}_{|u|\leq c}$, where $c \in (0, \infty)$ is another alternative for $g(\cdot)$; then $\rho_\tau(\cdot)$ is the M-quantile function.

Note that $g_3(\cdot)$ is a mixture of $g_1(\cdot)$ and $g_2(\cdot)$ which is more robust than $g_2(\cdot)$ and more efficient than $g_1(\cdot)$ (Kim and Oh, 2020). When $g(\cdot) = g_1(\cdot), g_2(\cdot), g_3(\cdot)$ in (3), $(\tilde{\boldsymbol{\alpha}}(\tau), \tilde{\boldsymbol{\beta}}(\tau))$ is the QRFE, ERFE and MQRFE estimate, respectively.

Remark 1. When $m_i = 1$ and $\alpha_i = 0$ for all i , QRFE becomes QR and is introduced by Koenker and Bassett Jr. (1978), ERFE is ER and is proposed by Newey and Powell (1987), and MQRFE is MQR and appears in (Breckling and Chambers, 1988).

As discussed previously, the weakness of fixed effects models is the large number of parameters to be estimated and the accumulation of estimation errors (You and Zhou, 2014). To overcome this drawback, Koenker (2004) uses the ideas of Tibshirani (1996) to penalize the intercepts by LASSO. Thus, the quantile estimate $(\hat{\boldsymbol{\alpha}}(\tau), \hat{\boldsymbol{\beta}}(\tau))$ of $(\boldsymbol{\alpha}(\tau), \boldsymbol{\beta}(\tau))$ solve the penalized version of (3),

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^n, \boldsymbol{\beta} \in \mathbb{R}^d} \left\{ \sum_{i=1}^n \sum_{j=1}^{m_i} \rho_\tau(Y_{ij} - \mathbf{X}_{ij}\boldsymbol{\beta} - \alpha_i) + \lambda \sum_{i=1}^n |\alpha_i| \right\}, \quad (4)$$

where λ is a tuning parameter. Let us consider

$$\boldsymbol{\varepsilon}(\tau) = \mathbf{Y} - \mathbf{X}\boldsymbol{\beta}(\tau) - \mathbf{Z}\boldsymbol{\alpha}(\tau), \quad (5)$$

which is estimated by $\tilde{\boldsymbol{\varepsilon}}(\tau) = \mathbf{Y} - \mathbf{X}\tilde{\boldsymbol{\beta}}(\tau) - \mathbf{Z}\tilde{\boldsymbol{\alpha}}(\tau)$, where $(\tilde{\boldsymbol{\alpha}}(\tau), \tilde{\boldsymbol{\beta}}(\tau))$ is a solution of (3). Following Koenker (2004) and Lamarche (2010), we take in (4) $\lambda = \tilde{\lambda} = \hat{\sigma}(\tilde{\boldsymbol{\varepsilon}}(0.5))/\hat{\sigma}(\tilde{\boldsymbol{\alpha}}(0.5))$, where $\hat{\sigma}(\mathbf{u})$ is the sample standard deviation of any vector $\mathbf{u} \in \mathbb{R}^N$. These authors propose to use $g_1(\cdot)$ in ρ_τ to solve (4), and here we extend this methodology by suggesting the two alternative loss functions $g_2(\cdot)$ and $g_3(\cdot)$, which conduce to PERFE and PMQRFE methods, respectively.

2.3 Asymptotic results

The convergence properties of QRFE and PQRFE estimators are established by Koenker (2004). Asymptotic normality of ERFE estimator is addressed by Barry et al. (2023). Here, we discuss the convergence of PERFE method.

Without loss of generality, we consider the case of balanced design experiment where $m_i = m$ for $i = 1, \dots, n$. The following assumptions are introduced.

(A1) The random vectors $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ are independent, and

$$\mathbb{V}[\boldsymbol{\Psi}_\tau(\boldsymbol{\varepsilon}_i(\tau))\boldsymbol{\varepsilon}_i(\tau)] = \mathbb{E}[\boldsymbol{\Psi}_\tau(\boldsymbol{\varepsilon}_i(\tau))\boldsymbol{\varepsilon}_i(\tau)\boldsymbol{\varepsilon}_i(\tau)^T\boldsymbol{\Psi}_\tau(\boldsymbol{\varepsilon}_i(\tau))] = \boldsymbol{\Sigma}_i(\tau),$$

where $\boldsymbol{\varepsilon}_i(\tau) = \mathbf{Y}_i - \mathbf{X}_i\boldsymbol{\beta}(\tau) - \mathbf{Z}_i\boldsymbol{\alpha}(\tau)$ and $\boldsymbol{\Psi}_\tau(\boldsymbol{\varepsilon}_i(\tau)) = \text{diag}(\psi_\tau(\varepsilon_{i,1}(\tau)), \dots, \psi_\tau(\varepsilon_{i,m}(\tau)))$.

(A2) The limiting forms of the following matrices are positive definite,

$$\mathbf{D}_0(\tau) = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{1}{m} \begin{bmatrix} \mathbf{Z}^T \boldsymbol{\Sigma}_\tau \mathbf{Z} & \mathbf{Z}^T \boldsymbol{\Sigma}_\tau \mathbf{X} / \sqrt{n} \\ \mathbf{X}^T \boldsymbol{\Sigma}_\tau \mathbf{Z} / \sqrt{n} & \mathbf{X}^T \boldsymbol{\Sigma}_\tau \mathbf{X} / n \end{bmatrix},$$

$$\mathbf{D}_1(\tau) = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{1}{m} \begin{bmatrix} \mathbf{Z}^T \mathbb{E}[\boldsymbol{\Psi}_\tau(\boldsymbol{\varepsilon}(\tau))] \mathbf{Z} & \mathbf{Z}^T \mathbb{E}[\boldsymbol{\Psi}_\tau(\boldsymbol{\varepsilon}(\tau))] \mathbf{X} / \sqrt{n} \\ \mathbf{X}^T \mathbb{E}[\boldsymbol{\Psi}_\tau(\boldsymbol{\varepsilon}(\tau))] \mathbf{Z} / \sqrt{n} & \mathbf{X}^T \mathbb{E}[\boldsymbol{\Psi}_\tau(\boldsymbol{\varepsilon}(\tau))] \mathbf{X} / n \end{bmatrix},$$

where $\Psi_\tau(\boldsymbol{\varepsilon}(\tau)) = \text{diag}(\psi_\tau(\varepsilon_{1,1}(\tau)), \dots, \psi_\tau(\varepsilon_{1,m}(\tau)), \dots, \psi_\tau(\varepsilon_{n,1}(\tau)), \dots, \psi_\tau(\varepsilon_{n,m}(\tau)))$ and $\Sigma_\tau = \mathbb{V}[\Psi_\tau(\boldsymbol{\varepsilon}(\tau))\boldsymbol{\varepsilon}(\tau)] = \text{diag}(\Sigma_1(\tau), \dots, \Sigma_n(\tau))$.

$$(A3) \quad \max_{i \geq 1, j \geq 1} \|\mathbf{X}_{ij}\| < M, \text{ where } \|\mathbf{X}_{ij}\|^2 = \mathbf{X}_{ij}\mathbf{X}_{ij}^T.$$

$$(A4) \quad \lim_{m \rightarrow \infty} \lambda_m / \sqrt{m} = \lambda_0 \geq 0.$$

Assumptions (A1)–(A3) are standard in longitudinal data analysis (Koenker, 2004; Barry et al., 2023). (A1) assumes independence between the subjects but allows dependency for each subject. When $\tau = 1/2$, $\psi_\tau(u) = 1/2$ and $\mathbf{D}_1(\tau)$ simplifies. (A4) appears in Knight and Fu (2000) to prove asymptotic normality of Lasso-type estimators.

Theorem 1. Under assumptions (A1)–(A4) and $\rho_\tau(u) = \psi_\tau(u)u^2$ in (4), $\sqrt{nm}(\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}(\tau))$ has the same limiting distribution as the component $\boldsymbol{\delta}_1$ of the minimiser of

$$V_0(\boldsymbol{\delta}) = \boldsymbol{\delta}^T \mathbf{D}_1(\tau) \boldsymbol{\delta} - 2\boldsymbol{\delta}^T \mathbf{B} + \lambda_0 \sum_{i=1}^n [\delta_{0,i} \text{sgn}(\alpha_i(\tau)) \mathbb{1}_{\alpha_i(\tau) \neq 0} + |\delta_{0,i}| \mathbb{1}_{\alpha_i(\tau) = 0}],$$

where $\boldsymbol{\delta} = (\boldsymbol{\delta}_0^T, \boldsymbol{\delta}_1^T)^T$, $\boldsymbol{\delta}_0 \in \mathbb{R}^n$, $\boldsymbol{\delta}_1 \in \mathbb{R}^d$, and \mathbf{B} has a $N(\mathbf{0}, \mathbf{D}_0(\tau))$ distribution.

Proof. We write $\rho_\tau(Y_{ij} - \mathbf{X}_{ij}\boldsymbol{\beta} - \alpha_i)$ as $\rho_\tau(Y_{ij} - \mu_{ij}(\tau) - \mathbf{X}_{ij}\boldsymbol{\delta}_1/\sqrt{nm} - \delta_{0,i}/\sqrt{m})$, where $\mu_{ij}(\tau) = Q_{Y_{ij}}(\tau|\mathbf{X}_{ij}) = \mathbf{X}_{ij}\boldsymbol{\beta}(\tau) + \alpha_i(\tau)$, $\boldsymbol{\delta}_1 = \sqrt{nm}(\boldsymbol{\beta} - \boldsymbol{\beta}(\tau))$ and $\delta_{0,i} = \sqrt{m}(\alpha_i - \alpha_i(\tau))$. For a given $\lambda = \lambda_m$, solving (4) is equivalent to minimize with respect to $\boldsymbol{\delta} = (\boldsymbol{\delta}_0^T, \boldsymbol{\delta}_1^T)^T$, $\boldsymbol{\delta}_0 \in \mathbb{R}^n$, $\boldsymbol{\delta}_1 \in \mathbb{R}^d$, the objective function $R_{nm}(\boldsymbol{\delta}) = R_{nm}^{(1)}(\boldsymbol{\delta}) + R_{nm}^{(2)}(\boldsymbol{\delta}_0)$, where

$$\begin{aligned} R_{nm}^{(1)}(\boldsymbol{\delta}) &= \sum_{i=1}^n \sum_{j=1}^m \rho_\tau(Y_{ij} - \mu_{ij}(\tau) - \mathbf{X}_{ij}\boldsymbol{\delta}_1/\sqrt{nm} - \delta_{0,i}/\sqrt{m}) - \rho_\tau(Y_{ij} - \mu_{ij}(\tau)), \\ R_{nm}^{(2)}(\boldsymbol{\delta}_0) &= \lambda_m \sum_{i=1}^n |\alpha_i(\tau) + \delta_{0,i}/\sqrt{m}| - |\alpha_i(\tau)|. \end{aligned} \tag{6}$$

The estimate

$$\hat{\boldsymbol{\delta}}(\tau) = \begin{pmatrix} \hat{\boldsymbol{\delta}}_0(\tau) \\ \hat{\boldsymbol{\delta}}_1(\tau) \end{pmatrix} = \begin{pmatrix} \sqrt{m}(\hat{\boldsymbol{\alpha}}(\tau) - \boldsymbol{\alpha}(\tau)) \\ \sqrt{nm}(\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}(\tau)) \end{pmatrix}$$

minimises $R_{nm}(\boldsymbol{\delta})$. Following Barry et al. (2023), under A(1)–A(3), the limiting form of $R_{nm}^{(1)}(\boldsymbol{\delta})$ is $\boldsymbol{\delta}^T \mathbf{D}_1(\tau) \boldsymbol{\delta} - 2\boldsymbol{\delta}^T \mathbf{B}$. Now, under (A4), $R_{nm}^{(2)}(\boldsymbol{\delta}_0) \rightarrow \lambda_0 \sum_{i=1}^n [\delta_{0,i} \text{sgn}(\alpha_i(\tau)) \mathbb{1}_{\alpha_i(\tau) \neq 0} + |\delta_{0,i}| \mathbb{1}_{\alpha_i(\tau) = 0}]$. Then, the limiting form of $R_{nm}(\boldsymbol{\delta})$ is $V_0(\boldsymbol{\delta})$ and the result follows from the convexity of $R_{nm}(\boldsymbol{\delta})$. \square

Remark 2. The asymptotic distributions of MQRFE and PMQRFE estimators are not discussed in this paper. It is expected that both estimators converge in distribution at the rate \sqrt{nm} to a Gaussian random vector. The asymptotic covariance matrix shall depend on parameter c in $g_3(\cdot)$ and be a balance of the results obtained for $g_1(\cdot)$ and $g_2(\cdot)$. A simple empirical study corroborates this insight as shown in Figure 1 as well as the empirical results discussed in Section 3. In Figure 1, the data are generated using Model (7) where U_{ij} follows a $N(0, 1)$ distribution. We plot the densities of $\sqrt{N}(\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}(\tau))$ for $\tau = 0.1, 0.5, 0.9$ for the three estimators PQRFE, PMQRFE, and PERFE. We observe that the PERFE (blue) estimator behaves similarly in the three scenarios. As we can see, the empirical distribution of PMQRFE is between the distributions of PQRFE and PERFE. A similar performance was observed for the empirical distribution of MQRFE.

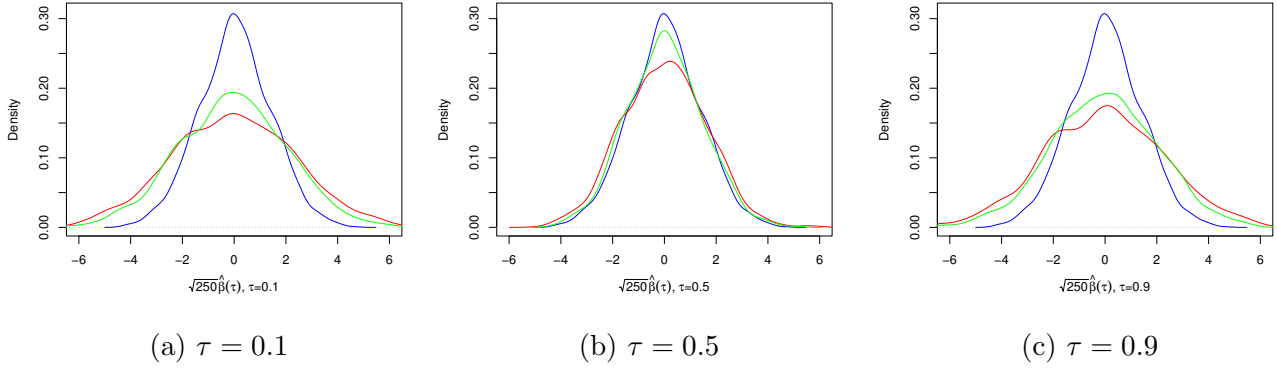


Figure 1: Densities of PQRFE (red), PMQRFE (green) and PERFE (blue) estimators for Model (7) ($\beta(\tau) = 0$).

3 Monte Carlo study

3.1 Monte Carlo design and measures of accuracy

In this section, a simulation study with a finite sample size is conducted to verify the performance of the estimators QRFE, PQRFE, MQRFE, PMQRFE, ERFE and PERFE under different scenarios, such as longitudinal data with homoscedasticity, heteroscedasticity and errors that follow Normal, heavy-tail and skew-symmetric distributions. These scenarios are quite common in many real problems.

The first model considered in this study is a simple case where the subjects α_i and covariates X_{ij} have only a location shift effect. This model is defined by

$$Y_{ij} = \alpha_i + X_{ij}\beta + U_{ij}, \quad (7)$$

and is denoted as location-shift model (LSM). The second model presents a location and scale effect. It is denoted by location-scale-shift model (LSSM) and is defined by

$$Y_{ij} = \alpha_i + X_{ij}\beta + (1 + X_{ij}\gamma)U_{ij}, \quad (8)$$

where $X_{ij} = W_i + V_{ij}$, W_i and V_{ij} are independent and follow a $N(0, 1)$ distribution. In (7) and (8), we consider the three following distributions:

- i) Normal case: $\alpha_i, U_{ij} \sim N(0, 1)$,
- ii) Student's t case: $\alpha_i, U_{ij} \sim t_3$,
- iii) Chi-squared case: $\alpha_i, U_{ij} \sim \chi_3^2$.

In all cases, we assume that $\beta = 0$ and $\gamma = 0.1$. For Model (7), we have $\beta(\tau) = 0$, and for Model (8), we have

$$\beta(\tau) = \begin{cases} \gamma Q_{U_{ij}}(\tau) & \text{if } 1 + X_{ij}\gamma > 0, \\ \gamma Q_{U_{ij}}(1 - \tau) & \text{if } 1 + X_{ij}\gamma < 0, \end{cases}$$

where $Q_{U_{ij}}(\tau)$ is the τ -quantile of U_{ij} . These experimental designs are originally proposed by Koenker (2004). The sample size follows a balanced design with $n = 50$, $m = 5$. Lamarche (2010) shows that the estimates of $\beta(\tau)$ are improved when n and m increase (less bias and variance), while $\tilde{\lambda}$ is more directly affected by m . Nine values of τ are considered, $\tau = 0.1, 0.2, \dots, 0.9$, extreme quantiles are progressively more challenging to estimate as far as they are from 0.5 (Wang et al.,

2012). The asymptotic extremes of MQR have already been investigated by Kokic et al. (2002); Kim and Oh (2020). In this context, we choose $c = 1$ since this value is also considered by these authors.

The parameter estimates, bias, root mean squared error (RMSE) and coverage probability (CP) with $100(1 - \delta)\%$ of the confidence interval are based on $S = 2,000$ replications and are given by

$$\text{i) Bias}(\hat{\beta}(\tau)) = \frac{1}{S} \sum_{s=1}^S (\hat{\beta}_s(\tau) - \beta(\tau)),$$

$$\text{ii) RMSE}(\hat{\beta}(\tau)) = \left(\frac{1}{S} \sum_{s=1}^S (\hat{\beta}_s(\tau) - \beta(\tau))^2 \right)^{1/2},$$

$$\text{iii) CP}(\hat{\beta}(\tau), \delta) = \frac{1}{S} \sum_{s=1}^S (\mathbf{1}_{\beta(\tau) \in [\hat{\beta}_s(\tau) \pm \widehat{\text{SE}}(\hat{\beta}_s(\tau)) \Phi^{-1}(1-\delta/2)]}),$$

where $\hat{\beta}_s(\tau)$ is the s th estimation of $\beta(\tau)$, Φ^{-1} is the inverse of the standard normal cumulative distribution, $\widehat{\text{SE}}(\hat{\beta}_s(\tau)) = (\widehat{\Sigma}_{n+1}/N)^{1/2}$ is the estimated standard error.

3.2 Empirical results

First, consider the estimation of λ . The optimal value suggested by Lamarche (2010) is the rate between the standard deviations of the vector \mathbf{U} and $\boldsymbol{\alpha}$, $\sigma(\mathbf{U})/\sigma(\boldsymbol{\alpha})$, see also, Section 3.3. Under the assumption that the distributions of \mathbf{U} and $\boldsymbol{\alpha}$ are the same for each case, $\lambda = 1$ is the optimal value. In this simulation, we consider the estimators QRFE, ERFE and MQRFE to estimate λ , and we fix $\tau = 0.5$. Normal, Student's t and Chi-squared distributions are considered for U_{ij} , $m = 5, n = 50$ and the plots in Figure 2 refer to the estimates over 2000 replications. This figure shows that all estimators tend to underestimate λ .

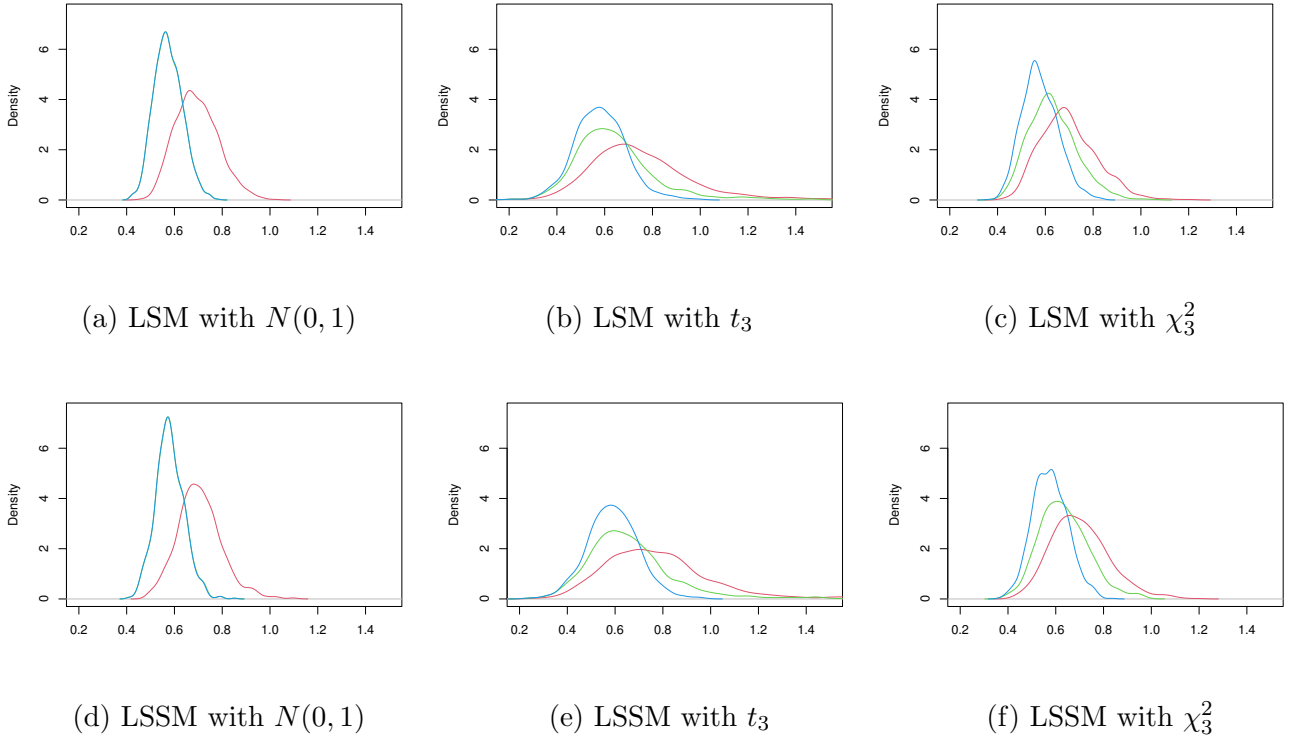


Figure 2: Density of λ estimated by QRFE (red), MQRFE (green) and ERFE (blue). Upper and lower rows display the estimates for Models (7) and (8), respectively.

Now, we compare the performance of the estimation methods for $\beta(\tau)$, $\tau = 0.1, 0.2, \dots, 0.9$. Figures 3 and 4 display the Bias, RMSE and empirical CP, for Models (7) and (8), respectively, with Normal, Student's t and Chi-squared distributions. Comparing Figures 3a, 3d, 3g with 4a, 4d, 4g, respectively, we see that the main difference between the two simulation models is related to the bias, which increases substantially in the case of Models (8).

Concerning RMSE and empirical CP, the performance of the methods changes according to the distributions. The ERFE and PERFE methods display, in general the best performance in terms of RMSE and CP, except in the case of a t_3 distribution where MQRFE (PMQRFE) methods present the best performance in terms of RMSE followed by QRFE (PQRFE) approaches, for both models. We observe that CP performance of the PQRFE is very unstable. This estimator substantially underestimates the true probability level for extreme values of τ . The estimators without penalization (dashed lines in the plots) reveal that the estimation of CP has the highest bias.

3.3 Computational considerations

All the computational techniques discussed in this paper are available in an R package (Danilevicz et al., 2022). We use RcppArmadillo to accelerate the calculations (Eddelbuettel and Sanderson, 2014). We have two objectives: obtaining a fair value of λ for the penalized methods and calculating the value of the covariance matrices of the $\beta(\tau)$ estimators for all procedures. To answer the first question, we use Lamarche (2010)'s statement that $\sigma(\mathbf{U})/\sigma(\boldsymbol{\alpha}) \rightarrow \lambda$ at least for $\tau = 0.5$. Both $\boldsymbol{\alpha}$ and \mathbf{U} are unknown, but we can achieve $\tilde{\boldsymbol{\alpha}}(0.5)$ and $\tilde{\boldsymbol{\varepsilon}}(0.5)$ by QRFE, ERFE and MQRFE, where $\tilde{\boldsymbol{\varepsilon}}(\tau) = (\tilde{\boldsymbol{\varepsilon}}_1(\tau), \dots, \tilde{\boldsymbol{\varepsilon}}_n(\tau))^T$, $\tilde{\boldsymbol{\varepsilon}}_i(\tau) = (\tilde{\varepsilon}_{i1}(\tau), \dots, \tilde{\varepsilon}_{im_i}(\tau))^T$ is the empirical residual equivalent to the τ -error from assumption (A1). Consequently, $\hat{\sigma}(\tilde{\boldsymbol{\varepsilon}}(0.5))/\hat{\sigma}(\tilde{\boldsymbol{\alpha}}(0.5))$ is a fair approximation of λ . Though, fixing λ as known, we can estimate $(\boldsymbol{\alpha}(\tau), \boldsymbol{\beta}(\tau))$ for PQRFE, PERFE and PMQRFE. This procedure dispenses much less computational effort than others like cross-validation (CV) or Bayesian information criterion (BIC), which require to reproduce the same calculations for a grid of possible values of λ . In Algorithm 1, we write a pseudo-code to explain how to achieve a PQRFE estimate, but it is analogous for PERFE and PMQRFE.

Data: \mathbf{Y}, \mathbf{X}

Result: Obtain $\hat{\boldsymbol{\beta}}(\tau)$ and $\hat{\boldsymbol{\alpha}}(\tau)$ by PQRFE.

while QRFE for $\tau = 0.5$ **do**

take initial values $\boldsymbol{\beta}(\tau)_0 = \mathbf{0}$ and $\boldsymbol{\alpha}(\tau)_0 = \mathbf{0}$
minimize (3)
obtain $\tilde{\lambda} = \hat{\sigma}(\tilde{\boldsymbol{\varepsilon}}(\tau))/\hat{\sigma}(\tilde{\boldsymbol{\alpha}}(\tau))$
return $\tilde{\boldsymbol{\beta}}(0.5)$, $\tilde{\boldsymbol{\alpha}}(0.5)$ and $\tilde{\lambda}$

end

while PQRFE **do**

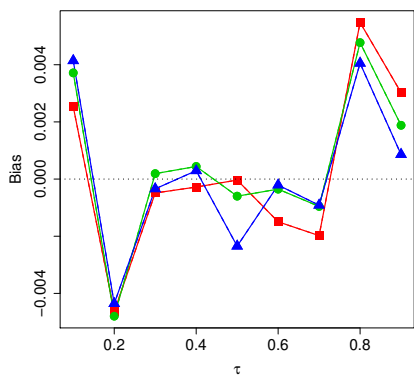
take initial values $\boldsymbol{\beta}(\tau)_0 = \tilde{\boldsymbol{\beta}}(0.5)$ and $\boldsymbol{\alpha}(\tau)_0 = \tilde{\boldsymbol{\alpha}}(0.5)$
minimize (4) under restriction $\lambda = \tilde{\lambda}$
return $\hat{\boldsymbol{\beta}}(\tau)$ and $\hat{\boldsymbol{\alpha}}(\tau)$

end

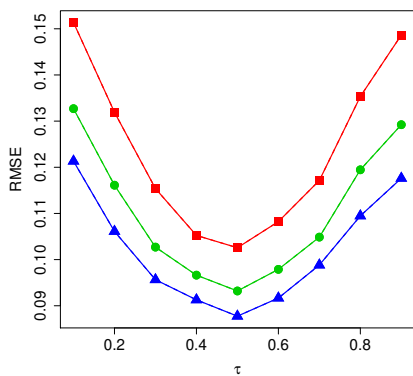
Algorithm 1: Example of optimizing by PQRFE.

4 Real data application

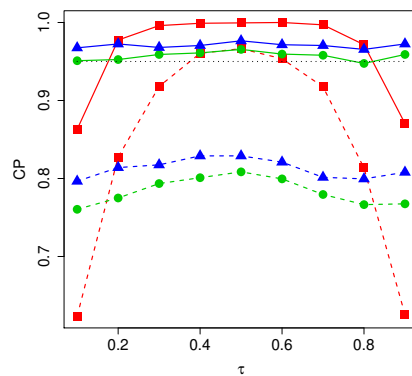
The country's ability to develop and maintain its social welfare and economic wealth depends on multiple factors. Having a consolidated high-tech sector and being able to export its technology with



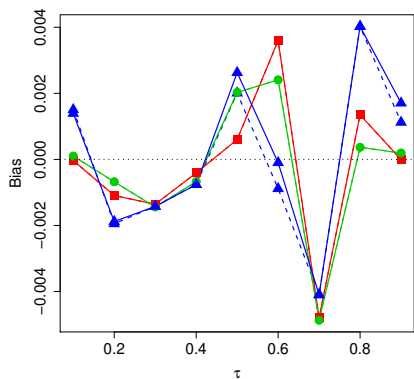
(a) Bias of LSM with $N(0,1)$



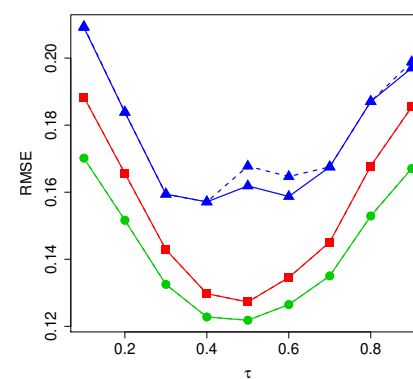
(b) RMSE of LSM with $N(0,1)$



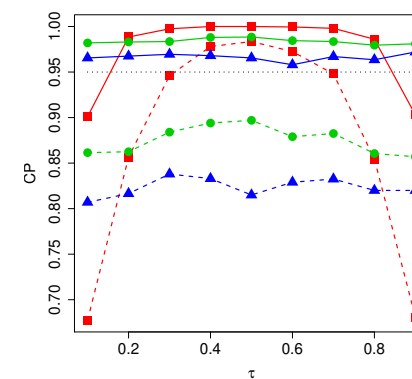
(c) CP of LSM with $N(0,1)$



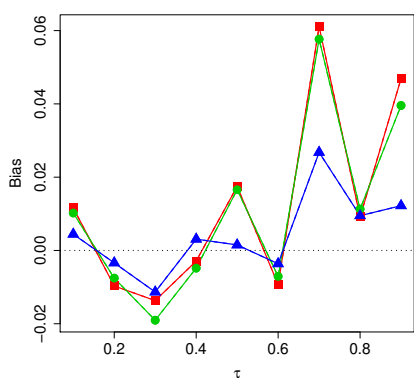
(d) Bias of LSM with t_3



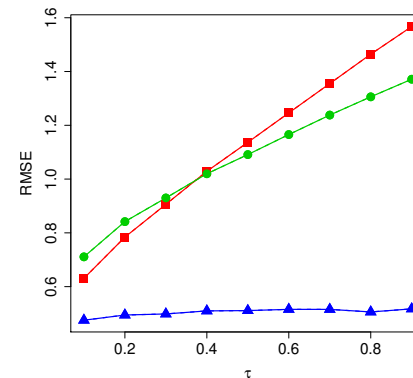
(e) RMSE of LSM with t_3



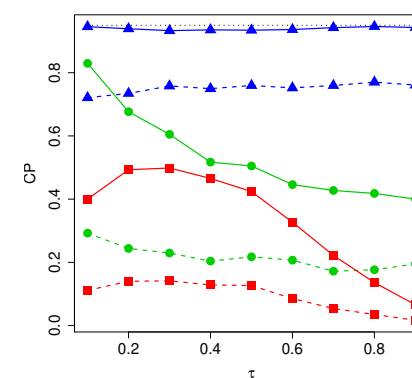
(f) CP of LSM with t_3



(g) Bias of LSM with χ_3^2

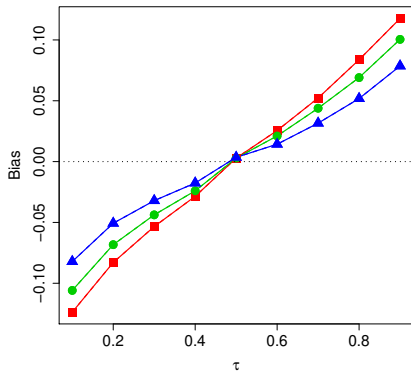


(h) RMSE of LSM with χ_3^2

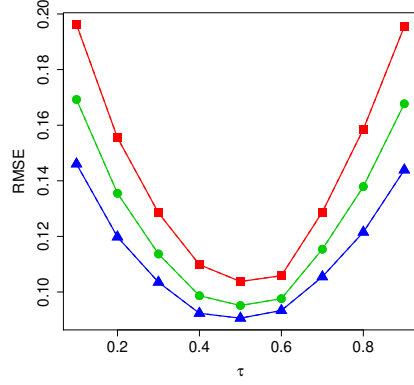


(i) CP of LSM with χ_3^2

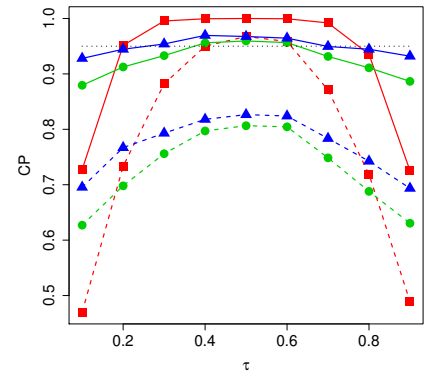
Figure 3: Bias, RMSE and CP with 95% of confidence of $\tilde{\beta}(\tau)$ estimate by QRFE (dashed line with red square), MQRFE (dashed line with green circle), ERFE (dashed line with blue triangle) and $\hat{\beta}(\tau)$ estimate by PQRFE (solid line with red square), PMQRFE (solid line with green circle), PERFE (solid line with blue triangle). Results for Model 7.



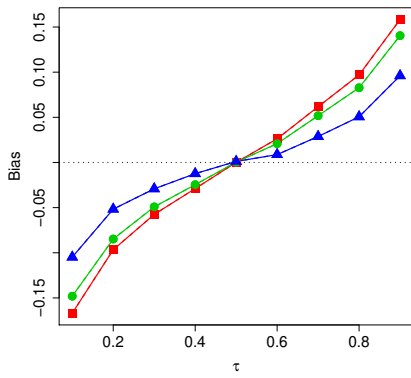
(a) Bias of LSSM with $N(0, 1)$



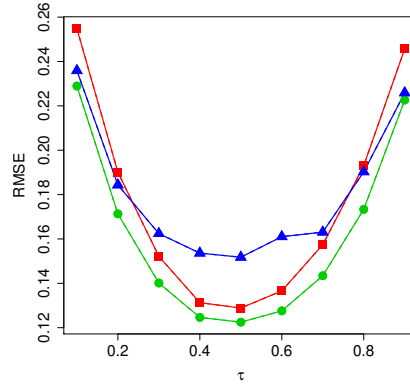
(b) RMSE of LSSM with $N(0, 1)$



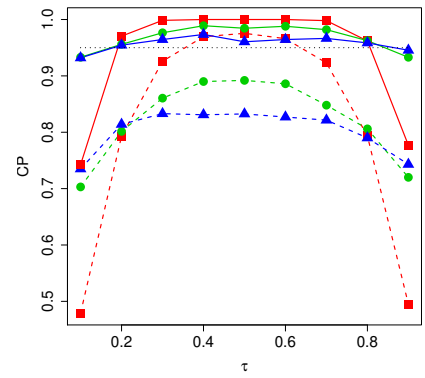
(c) CP of LSSM with $N(0, 1)$



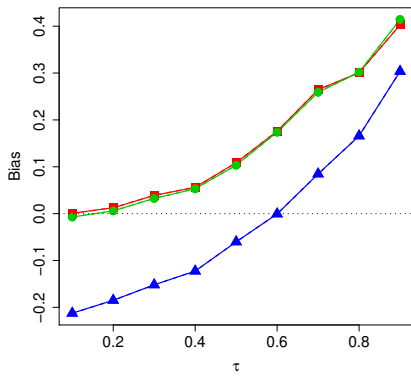
(d) Bias of LSSM with t_3



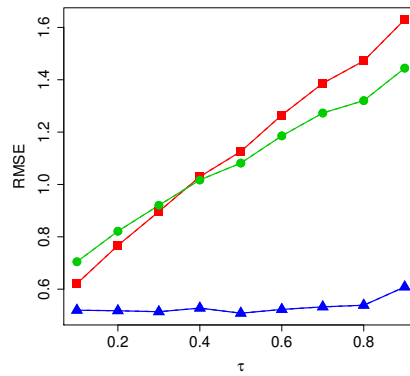
(e) RMSE of LSSM with t_3



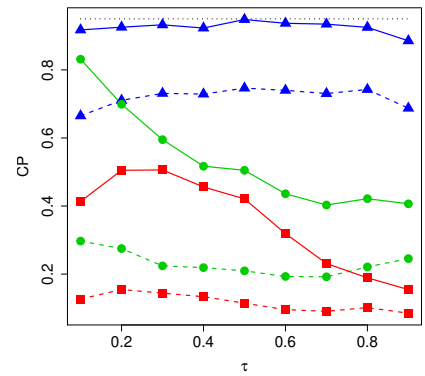
(f) CP of LSSM with t_3



(g) Bias of LSSM with χ_3^2



(h) RMSE of LSSM with χ_3^2



(i) CP of LSSM with χ_3^2

Figure 4: Bias, RMSE and CP with 95% of confidence of $\tilde{\beta}(\tau)$ estimate by QRFE (dashed line with red square), MQRFE (dashed line with green circle), ERFE (dashed line with blue triangle) and $\hat{\beta}(\tau)$ estimate by PQRFE (solid line with red square), PMQRFE (solid line with green circle), PERFE (solid line with blue triangle). Results for Model 8.

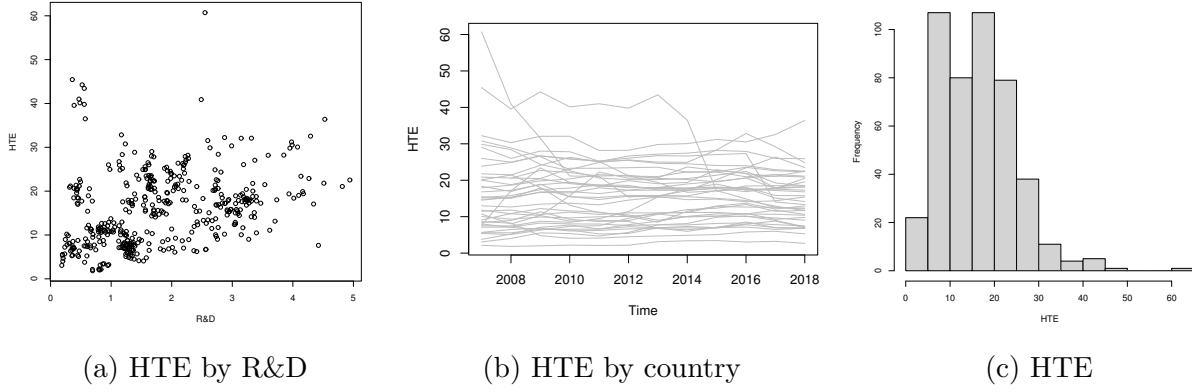


Figure 5: Exploratory data analysis

high-added value is one of these factors (Turen and Gökmen, 2013). Increasing high-tech exports has a direct and significant implication on increasing the GDP (Yoo, 2008; Falk, 2009). However, achieving and maintaining high levels of innovation and competitiveness may only be achieved with public policies that guide and support an innovation-friendly strategy (Baesu et al., 2015). One way of measuring a country’s commitment to its high technological sector is through its investments in the base of this production chain, such as research and technology (Alemu, 2012).

The data is available in the World Bank¹. The response variable is high-technology exports (% of manufactured exports) (HTE) and the predicted variable is research and development expenditure (% of GDP) (R&D) (Alemu, 2012; Turen and Gökmen, 2013; Baesu et al., 2015). The data is yearly recorded of 38 OECD countries from 2007 to 2019. The data has 26 missing observations and the model designed is unbalanced with $m_i = 4, 6, 10, 11, 12$, corresponding to a sample of size $N = 430$.

First, we perform a brief exploratory data analysis. Figure 5a displays a scattering pattern opening in a cone, a typical heteroscedastic phenomenon in this data type. Thus, quantile methods are more suitable than standard average procedures since the former can better capture the behavior that changes from one stratum to another. Figure 5b shows the trajectory of each country, there is a large dispersion between them. Clearly, each state has a stable location which may be shared with other countries in some cases (mainly in the case of the states with the lowest values of HTE). This suggests a longitudinal model with different α_i ’s. Figure 5c shows that the response variable is asymmetric, and this property of the data discourages the use of methods such as QRFE, PQRFE, MQRFE and PMQRFE since, based on the simulation study, these approaches appear to be very sensitive to the skewness property of the empirical distribution. Additionally, the non-parametric statistical one-sample normality Kolmogorov-Smirnov and Breusch-Pagan tests are also performed. As expected, these tests strongly indicated that the data is not normal and non homoscedastic, respectively.

Based on the above discussion, we fit the following model to this data,

$$Y_{ij} = \alpha_i + \beta \text{R\&D}_{ij} + U_{ij}, \quad (9)$$

where, for $i = 1, \dots, 38$ and $j = 1, \dots, 12$, Y_{ij} is the HTE output of i th country at j th year, R\&D_{ij} is the yearly national investment on research, α_i is the country effect and U_{ij} is the measurement error of the model. Model (9) is fitted by PQRFE, PMQRFE and PERFE methods, since according to Section 3, the empirical CP performance of these methods is better than the other approaches.

Figure 6 shows the central fit of the estimates of $\beta(\tau)$ and their empirical confidence intervals. This figure displays an upward behavior of $\hat{\beta}(\tau)$ as τ increases. The estimates $\hat{\beta}(\tau)$ obtained with

¹Collected on May 26, 2022

PMQRFE and PERFE methods are significantly different from zero for all values of τ , while the estimate of $\beta(\tau)$ using PQRFE method is non-zero only if $\tau > 0.5$.

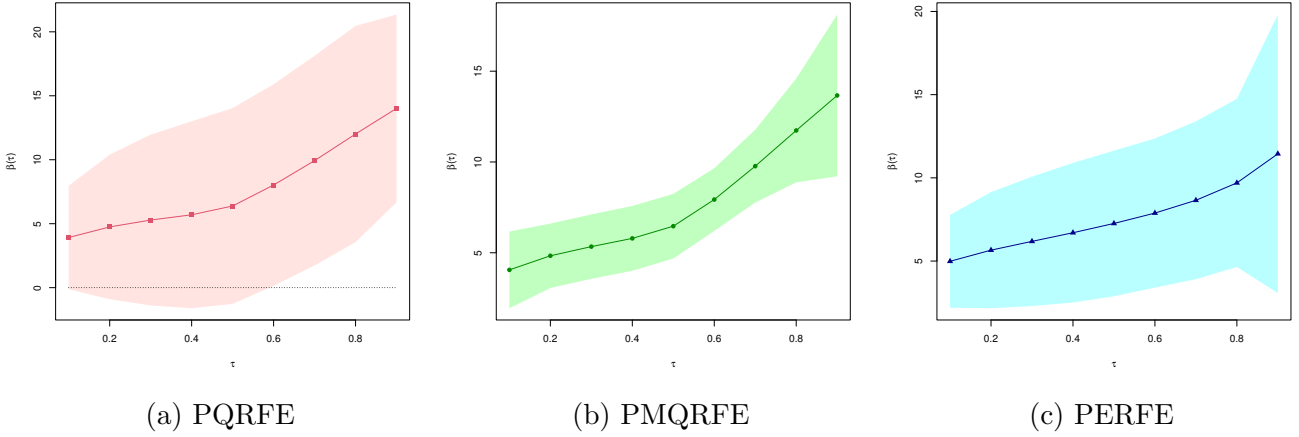


Figure 6: R&D coefficient $\hat{\beta}(\tau)$ and its confidence interval for $\tau \in [0.1, 0.9]$

Figure 7 shows the $\hat{\beta}(\tau)$ R&D_{ij} lines for $\tau = 0.1, 0.9$ with their confidence intervals. These lines display a cone shape, which is not surprising since the standard deviation of the predicted variable appears to be non-constant. This corroborates the use of quantile methods to fit the data. The PMQRFE and PERFE approaches led to estimated models, which appear to be more accurate than the PQRFE method. For all values of τ , R&D investments positively impact HTE. We conclude that R&D constitutes an investment that influences the composition of a mature economy centred on innovation, whose export of high-tech products does not exclusively depend on commodities.

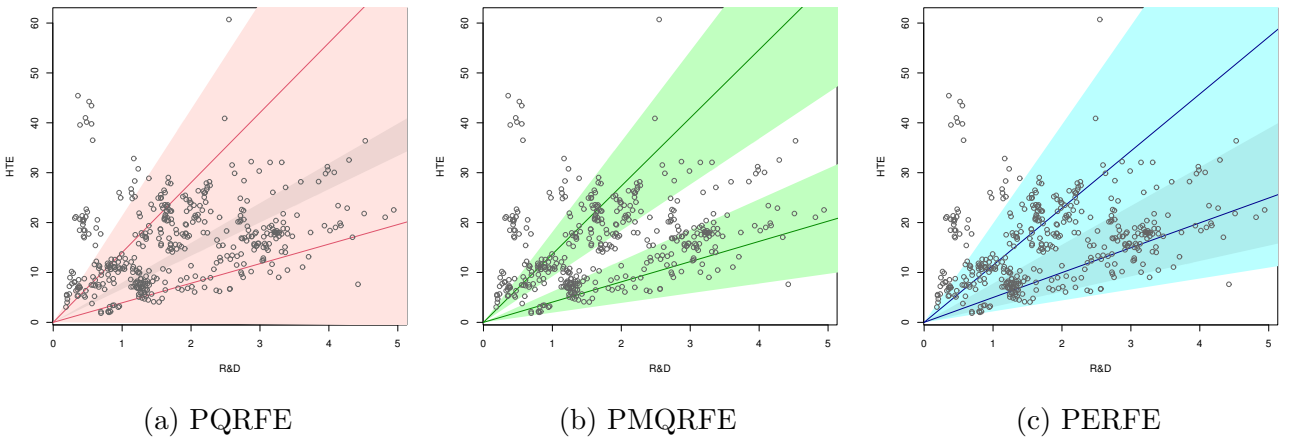


Figure 7: $\hat{\beta}(\tau)$ R&D_{ij} lines and confidence intervals for $\tau = 0.1, 0.9$

5 Conclusion

This paper suggests alternatives to the check function in the quantile regression to estimate the parameters of fixed effect models. The new approaches are named PERFE, MQRFE, and PMQRFE. The finite sample investigation indicates that the most accurate estimation method is PERFE when the errors have a Gaussian distribution. In contrast, PMQRFE is recommended for a symmetric heavy-tailed distribution, and PERFE is preferable when the distribution is skewed. Financial panel

data are analyzed where HTE and R&D are the response and predicted variables, respectively. The distribution is skewed, and the PERFE method shows that R&D has a positive influence on HTE, which is in line with the economics literature.

References

- Alemu, A. M. (2012). The Effect of R&D on High-Tech Product Export Competitiveness: Empirical Evidence from Panel Data of East Asian Economies. *STI Policy Review*, 3(1):46–62.
- Alfö, M., Marino, M. F., Ranalli, M. G., Salvati, N., and Tzavidis, N. (2021). M-quantile regression for multivariate longitudinal data with an application to the Millennium Cohort Study. *J. R. Stat. Soc., C*, 70(1):122–146.
- Baesu, V., Albuлесcu, C. T., Farkas, Z.-B., and Draghici, A. (2015). Determinants of the High-tech Sector Innovation Performance in the European Union: A Review. *Proc. Technol.*, 19:371–378. 8th International Conference Interdisciplinarity in Engineering, INTER-ENG 2014, 9-10 October 2014, Tirgu Mures, Romania.
- Barry, A., Oualkacha, K., and Charpentier, A. (2023). Alternative fixed-effects panel model using weighted asymmetric least squares regression. *Stat. Methods Appl.*, 32(3):819–841.
- Breckling, J. and Chambers, R. (1988). M-quantiles. *Biometrika*, 75(4):761–771.
- Breckling, J., Kokic, P., and Lubke, O. (2001). A note on multivariate M-quantiles. *Stat. Probab. Lett.*, 55:39–44.
- Danilevicz, I. M., Reisen, V. A., and Bondon, P. (2022). *pqrfe: Penalized Quantile Regression with Fixed Effects*. R package version 1.1.
- Diggle, P. (1988). An Approach to the Analysis of Repeated Measurements. *Biometrics*, 44(4):959–971.
- Eddelbuettel, D. and Sanderson, C. (2014). RcppArmadillo: Accelerating R with high-performance C++ linear algebra. *Comput. Stat. Data Anal.*, 71:1054–1063.
- Falk, M. (2009). High-tech exports and economic growth in industrialized countries. *Appl Econ Lett*, 16(10):1025–1028.
- Galarza, C., Castro, L., Louzada, F., and Lachos, V. (2020). Quantile regression for nonlinear mixed effects models: a likelihood based perspective. *Stat. Pap.*, 61:1281–1307.
- Galvão, A. F., Gu, J., and Volgushev, S. (2020). On the unbiased asymptotic normality of quantile regression with fixed effects. *J. Econom.*, 218(1):178–215.
- Geraci, M. and Bottai, M. (2007). Quantile regression for longitudinal data using the asymmetric Laplace distribution. *Biostatistics*, 8(1):140–154.
- Geraci, M. and Bottai, M. (2014). Linear quantile mixed models. *Stat. Comput.*, 24:461–479.
- Horowitz, J. L. and Lee, S. S. (2004). Semiparametric estimation of a panel data proportional hazards model with fixed effects. *J. Econom.*, 119(1):155–198.
- Huber, P. J. (1964). Robust Estimation of a Location Parameter. *Ann. Stat.*, 1(53):73–101.

- Jones, M. C. (1994). Expectiles and M-quantiles are quantiles. *Stat. Probab. Lett.*, 20(2):149–153.
- Kim, J. and Oh, H.-S. (2020). Pseudo-quantile functional data clustering. *J Multivar Anal*, 178:104626.
- Knight, K. and Fu, W. (2000). Asymptotics for Lasso-Type Estimators. *Ann. Stat.*, 28:1356–1378.
- Koenker, R. (2004). Quantile regression for longitudinal data. *J. Multivar. Anal.*, 91(1):74–89.
- Koenker, R. (2005). *Quantile Regression*. Cambridge University Press.
- Koenker, R. and Bassett Jr., G. (1978). Regression quantiles. *Econometrica*, 46:33–50.
- Kokic, P., Breckling, J., and Lübke, O. (2002). A New Definition of Multivariate M-quantiles. In Dodge, Y., editor, *Statistical Data Analysis Based on the L1-Norm and Related Methods*, pages 15–24, Basel. Birkhäuser Basel.
- Laird, N. M. and Ware, J. H. (1982). Random-Effects Models for Longitudinal Data. *Biometrics*, 38(4):963–974.
- Lamarche, C. (2010). Robust penalized quantile regression estimation for panel data. *J. Econom.*, 157(2):396–408.
- Lamarche, C. and Parker, T. (2021). Wild Bootstrap Inference for Penalized Quantile Regression for Longitudinal Data. working paper or preprint.
- Li, D., Wang, L., and Zhao, W. (2022). Estimation and inference for multikink expectile regression with longitudinal data. *Stat. Med.*, 41(7):1296–1313.
- Li, H., Liu, Y., and Luo, Y. (2020). Double Penalized Quantile Regression for the Linear Mixed Effects Model. *J. Syst. Sci. Complex.*, 33:2080–2102.
- Marino, M. and Farcomeni, A. (2015). Linear quantile regression models for longitudinal experiments: An overview. *Metron*, 73:229–247.
- Newey, W. and Powell, J. (1987). Asymmetric Least Squares Estimation and Testing. *Econometrica*, 55(4):819–847.
- Schnabel, S. and Eilers, P. (2009). Optimal expectile smoothing. *Comput. Stat. Data Anal.*, 53:4168–4177.
- Tian, Y., Wang, L., Tang, M., and Tian, M. (2021). Weighted composite quantile regression for longitudinal mixed effects models with application to AIDS studies. *Commun. Stat. Simul. Comput.*, 50(6):1837–1853.
- Tibshirani, R. (1996). Regression Shrinkage and Selection via the Lasso. *J. R. Stat. Soc., B*, 58(1):267–288.
- Turen, U. and Gökmen, Y. (2013). The determinants of high technology exports volume: A panel data analysis of eu-15 countries. *Int. J. Econ. Manag. Soc. Sci.*, 2(3):217–232.
- Tzavidis, N., Salvati, N., Schmid, T., Flouris, E., and Midouhas, E. (2016). Longitudinal analysis of the strengths and difficulties questionnaire scores of the Millennium Cohort Study children in England using M-quantile random-effects regression. *J. R. Stat. Soc., A*, 179(2):427–452.

- Waltrup, L. S. and Kauermann, G. (2017). Smooth expectiles for panel data using penalized splines. *Stat Comput*, 27:271–282.
- Waltrup, L. S., Sobotka, F., Kneib, T., and Kauermann, G. (2015). Expectile and quantile regression - David and Goliath? *Stat. Modelling*, 15(5):433–456.
- Wang, H. J., Li, D., and He, X. (2012). Estimation of High Conditional Quantiles for Heavy-Tailed Distributions. *J. Am. Stat. Assoc.*, 107(500):1453–1464.
- Yoo, S.-H. (2008). High-technology exports and economic output: an empirical investigation. *Appl Econ Lett*, 15(7):523–525.
- You, J. and Zhou, X. (2014). Asymptotic theory in fixed effects panel data seemingly unrelated partially linear regression models. *Econ. Theory*, 30(2):407–435.

Declarations

Conflicts of interest

The authors declare no conflicts of interest.

Acknowledgments

The authors thank two anonymous reviewers for their valuable comments to improve the quality of the paper.

Funding

Part of this paper was written when Valdério Anselmo Reisen was visiting CentraleSupélec. This author is indebted to CentraleSupélec, CNPq and FAPES for their financial support. Ian Meneghel Danilevicz is indebted to Université Paris-Saclay and CAPES for their financial support. This research was also supported by DATAIA Convergence Institute as part of the *Programme d'Investissement d'Avenir* (ANR17-CONV-0003) operated by Université Paris-Saclay.